Faculty V
Institute of Mechanics

Numerical examination of the state of deformation in large terrestrial objects

«Bachelor's Thesis »

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Contents

1 Summary

The reader could find in this paper an introduction to a numerical examination of the state of deformation of large bodies such as planets. This numerical study is done under the continuum mechanics theory and solved following the Finite Elements Method using the code-free FEniCS [1].
This study is elaborated under two different hypothesis. The first one considers the Earth as a large object and therefore the solution is found under the hypothesis of finite displacements. With this consideration, the body is studied in the current placement in which are used the Almansi's strains notation. Moreover, even though the solution to be found depends on the displacement, the model used to solve it can be a new variable called beta (β) that make the convergence of the solution easier.
The second hypothesis used to solve the problem is viscoelasticity. Therefore, the solution now depends on time apart form the position. Nonetheless, before starting the 3-dimensional case of the Earth, some other examples are done in order to get used to work under viscoelasticity. Consequently, the study in 1D and in 3D of a viscoelastic beam can also be found in the present paper. As a matter of time and excessive difficulty, the study of the 3D Earth under viscoelastic hypothesis is done under infinitesimal strains.
Both studies and previous cases are a satisfying approximation to examine the deformation state both in beams and massive bodies such as planets.

2 Introduction

This paper contains the memories of the work made by the author in the Department of Mechanics of the Technische Universität Berlin from the 27th of February of 2015 until the 4th of August of 2015. It is, hence, a summary of the learning and progression made in the bachelor's thesis.

2.1 Objectives

The bachelor's thesis means the finalization of four years of learnings in my case of Engineering. Therefore, the main objective of this report is to summarize these learnings in a thesis worth reading. Moreover, in this process I expect to become familiar with general competences such as: writing of reports, the use of new engineering tools and how to perform a satisfying oral presentation, amongst others.
Referring to the specific aim of the bachelor's thesis itself, there is one that can be described as the main objective, which is the following:

- Perform a numerical study of the state of deformation of large terrestrial objects and more concretely of the Earth. It is studied under two different models: The first one uses a linear elastic model of the material and the second is studied under the hypothesis of the viscoelasticity.

Furthermore, there are other secondary objectives which need to be achieved in order to be able to fulfill the main one.

- Reaching a high enough level of the new software used, which solves differential equations.
- Acquire the knowledge of Continuum Mechanics required for the study.
- Check some mathematical basic tools such as transfer of coordinate systems, manipulation with tensors and resolution of simple differential equations.
- Perform previous studies with several different simplifications in the model to gain experience in the field.

3 Getting to know the subject and the tools

The whole academic project turns around the study of the Earth as a self-gravitating object inside which exist stresses and strains still to be studied. Thus, the aim of this project is to determine how big are these strains, stresses, deformations, displacements... Since the project is driven by the department of Continuum Mechanics, the Earth is studied as a continuum body. The Earth will be considered as a solid, with a homogeneous distribution of density, made only by one material.
On the first week of work, it was needed to achieve a certain knowledge in the continuum theory and the mathematical tools implicitly used in it. The book of Prof. Dr. rer. nat. Wolfgang H. Müller „An expedition to continuum theory“ was the main reference used [12]. Furthermore, there are more articles about the flattening study of the Earth worth reading to get to know the State of Art [3, 4, 9, 10, 13]. Hence, the first days of the study were invested in reading and taking notes of the bibliography detailed above.
The solution of the problem considered needs of differential equations to be solved. Because of the complexity of such equations, FEniCS (a free-code program which solves differential equations by Finite Element Methods) is being used.

3.1 State of art

3.1.1 Earth Models

The object of study is a celestial body and more concretely the planet Earth. As a simplification, it will be considered as a perfect sphere, still knowing that it is not completely true.
As for its material properties (POISSON's Coefficient (ν), YOUNG's Modulus (E) and density (ρ) amongst others) there is only one hypothesis considered:

- Isotropic sphere of homogeneous density including both the surface and the core.

Moreover, the previous hypothesis is considered as spherically symmetrical (not depending on the polar nor the azimuthal axis) [12, §9.6].

3.1.2 Previous calculations

LOVE's approach A. E. H. LOVE solves the problem of a homogeneous isotropic sphere held strained by the mutual gravitation of his own body in 1927. He showed that the radial strain is contraction within the surface

$$r = r_s \left(\frac{3-\nu}{3+3\nu} \right)^{\frac{1}{2}} , \tag{3.1}$$

but it is extension outside this surface, where r_s is the radius of the sphere and ν is POISSON's ratio. But unfortunately this result cannot be applied to the case of earth, since the strain inside the Earth is so great that it cannot be studied under the small strain theory.
Nonetheless, according to this theory, the actual radius for the Earth in which the strain changes of sign should be around

$$r = r_s \left(\frac{3-\nu}{3+3\nu} \right)^{\frac{1}{2}} = r_s \left(\frac{3-0,3}{3+3\cdot 0,3} \right)^{\frac{1}{2}} = 0,832 r_s \, , \tag{3.2}$$

considering the POISSON's ratio (ν) as the one from the iron.

3.2 FEniCS

FEniCS is a software that needs to be run in one of the following programming languages: C++ or Python. When having a minimal knowledge in Python it is not difficult to understand the examples/tutorials found in the website of FEniCS Project [1]. By reading and running them you are able to get to know the software quite well.

3.2.1 Previous Study: Linear elastic 3D Earth model

With the help of Mr. Lofink, a numerical solution to a first problem war reached. In it with set out a 3D Earth model in which the earth was a sphere made out of steel with a homogeneous distribution of density. The model's behavior was linear elastic and the mesh was made of homogeneous triangles. There were three variables of the solution to be studied: the potential, the gravitation and the displacement.

At the beginning, calculus were made on the model taking into account only gravitational forces. To reach the solution, the local balance of linear momentum was used under the hypothesis of a stationary model.
Afterwards, the forces caused by the rotation were added to the model. When the rotative forces were taken into consideration, the results did not change excessively compared with the previous solution. Therefore, the rotative forces could be underestimated, in order to simplify our model.
After knowing how the software worked, Mr. Lofink sat an exercise: to improve the boundary conditions, it was needed to refine the mesh only in the core of the sphere. After some struggles the task was completed successfully. Although the results varied in the third significant figure (no more than 1% of the previous solution), the time of computation had increased more than four times. Therefore, the conclusion reached was that there was not a significant difference between the two different meshes.

3.2.2 Get to know FEniCS

After a first glance on FEniCS, Mr. Lofink suggested to study the tutorials on FEniCS web page in order to learn the basics. Not only by reading it, but also by writing simple code described on the tutorial, you can learn the four basic steps that you have to follow when solving a Partial Differential Equation (PDEs) in FEniCS [1]:

1. Identify the PDE and its boundary conditions.
2. Reformulate the PDE problem as a variational problem.
3. Make a Python program where the formulas in the variational problem are coded, along with definitions of input data and variables such as the position (x) and the displacement (u). A mesh for the spatial domain will be also needed.
4. Add statements in the program for solving the variational problem and visualizing the results.

Moreover, there are multiple examples to practice different kind of solvers (linear, non-linear, preconditioner...), different kinds of boundary conditions (Dirichlet and Neumann, amongst others), different ways of implementing the same equation, etc. Nonetheless, it must be said that this tutorial was mainly a resource to look at when I had questions about how to continue in order to solve the problem provided. The examples in the tutorial and previous example 3.2.1 were used as references to help me solve the actual code.

4 Current placement

4.1 Displacement Formulation

We started working in a one-dimensional model of the Earth in which its elastic material properties are constant and homogeneous and equal to the ones of the steel. Nonetheless, the density of the Earth is the one resulting of dividing the mass (m_{Earth}) by the volume (V_{Earth}) of the Earth nowadays. Therefore

$$\rho_{\text{E}} = \frac{m_{\text{Earth}}}{V_{\text{Earth}}} = \frac{m_{\text{Earth}}}{\frac{4}{3}\pi a^3} = \frac{m_{\text{Earth}}}{\frac{4}{3}\pi a^3} = \frac{5.972 \cdot 10^{24}\,\text{kg}}{\frac{4}{3}\pi\left(6.3781 \cdot 10^{7}\right)^3\ \text{m}^3} = 5515\,\text{kg/m}^3\,, \tag{4.1}$$

where a stands for the outer radius of the Earth nowadays.
The terrestrial body is described as a linear elastic isotropic self gravitational object completely isolated of other bodies. Moreover, we only take into consideration self gravitation forces and we do not consider rotation. Hence, we only study the radial component, because both azimuthal and polar angles would be meaningless, according to the results from the previous example 3.2.1. The solution to the problem will be the displacement $\boldsymbol{u}$ along the radial axis.

4.1.1 Variables and Equations

In the first place, we need to reach the expressions of stresses in the diagonal (σ_{rr}, $\sigma_{\vartheta\vartheta}$and $\sigma_{\varphi\varphi}$) calling the constitutive law:

$$\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{e})\, \mathbf{1} + 2\mu \boldsymbol{e}\,, \tag{4.2}$$

where λ is the first and μ is the second LAMÉ parameter and $\boldsymbol{e}$ stands for the ALMANSI's finite strain tensor because we assume that the strains are being calculated under the hypothesis of finite deformation.
Assuming the transformation from reference to current placement in spherical coordinates:

$$\begin{aligned} r &= R + u_r(r) \\ \vartheta &= \Theta \\ \varphi &= \Phi \end{aligned} \quad . \tag{4.3}$$

In the previous expression (4.3) R is the radius in the referential placement, $u_r(r)$ is the displacement in the r-direction and both ϑ and φ are the polar and azimuthal angles respectively.
After calculating the deformation gradient for this case:

$$\boldsymbol{F} = \begin{pmatrix} \frac{\mathrm{d}r}{\mathrm{d}R} & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & \frac{r}{R} \end{pmatrix}, \tag{4.4}$$

we can calculate the expression of two times ALMANSI's geometrically non-linear strains as follows:

$$2\boldsymbol{e} = \left(\mathbf{1} - \boldsymbol{B}^{-1}\right) = \left(\mathbf{1} - \left(\boldsymbol{F}\cdot\boldsymbol{F}^{\mathrm{T}}\right)^{-1}\right), \tag{4.5}$$

and it results:

$$2\boldsymbol{e} = \begin{pmatrix} 1 - \left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2 & 0 & 0 \\ 0 & 1 - \left(\frac{R}{r}\right)^2 & 0 \\ 0 & 0 & 1 - \left(\frac{R}{r}\right)^2 \end{pmatrix}. \tag{4.6}$$

Applying these changes

$$\begin{aligned} r = R + u_r \quad & \Rightarrow u_r = r - R \quad \Rightarrow \frac{R}{r} = 1 - \frac{u_r}{r} \\ & \Rightarrow \frac{\mathrm{d}u_r}{\mathrm{d}r} = 1 - \frac{\mathrm{d}R}{\mathrm{d}r} \end{aligned} \tag{4.7}$$

we arrive to the expression

$$
\begin{aligned}
2\boldsymbol{e} &= \begin{pmatrix} 1-\left(1-\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)^2 & 0 & 0 \\ 0 & 1-\left(1-\frac{u_r}{r}\right)^2 & 0 \\ 0 & 0 & 1-\left(1-\frac{u_r}{r}\right)^2 \end{pmatrix} = \\
&= \begin{pmatrix} \frac{\mathrm{d}u_r}{\mathrm{d}r}\left(1-\frac{1}{2}\frac{\mathrm{d}u_r}{\mathrm{d}r}\right) & 0 & 0 \\ 0 & \frac{u_r}{r}\left(1-\frac{1}{2}\frac{u_r}{r}\right) & 0 \\ 0 & 0 & \frac{u_r}{r}\left(1-\frac{1}{2}\frac{u_r}{r}\right) \end{pmatrix}.
\end{aligned}
\tag{4.8}
$$

The expressions for the components of the strain tensor can be simplified as follows:

$$
\begin{aligned}
e_{rr} &= \frac{1}{2}\left[1-\left(1-\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)^2\right] = \frac{1}{2}\left[1-\left(1-2\frac{\mathrm{d}u_r}{\mathrm{d}r}+\left(\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)^2\right)\right] \\
&= \frac{\mathrm{d}u_r}{\mathrm{d}r}-\frac{1}{2}\left(\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)^2 = u_r'\left(1-\frac{1}{2}\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)
\end{aligned}
\tag{4.9}
$$

and

$$
e_{\vartheta\vartheta} = e_{\varphi\varphi} = \frac{1}{2}\left[1-\left(1-\frac{u_r}{r}\right)^2\right] = \frac{1}{2}\left[1-\left(1-2\frac{u_r}{r}+\left(\frac{u_r}{r}\right)^2\right)\right] = \frac{u_r}{r}\left(1-\frac{1}{2}\frac{u_r}{r}\right). \tag{4.10}
$$

The next step is calculating the trace of the ALMANSI's tensor $\boldsymbol{e}$, bearing in mind that the described in the expression 4.8 is two times this tensor. Therefore it stays as follows:

$$
\operatorname{tr}(\boldsymbol{e}) = \frac{\mathrm{d}u_r}{\mathrm{d}r}\left(1-\frac{\frac{\mathrm{d}u_r}{\mathrm{d}r}}{2}\right)+\frac{2u_r}{r}\left(1-\frac{1}{2}\frac{u_r}{r}\right). \tag{4.11}
$$

With the previous 3 steps (4.9, 4.10 and 4.11), it can be calculated the expression of stress in the three directions of the axis. Therefore, we can take out the whole stress tensor.

$$
\begin{aligned}
\sigma_{rr} &= \lambda \operatorname{tr}(\boldsymbol{e}) + 2\mu e_{rr} = \lambda\left[\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\left(1-\tfrac{1}{2}\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\right)+\tfrac{2u_r}{r}\left(1-\tfrac{1}{2}\tfrac{u_r}{r}\right)\right]+2\mu\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\left(1-\tfrac{1}{2}\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\right) \\
&= (\lambda+2\mu)\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\left(1-\tfrac{1}{2}\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\right)+2\lambda\tfrac{u_r}{r}\left(1-\tfrac{1}{2}\tfrac{u_r}{r}\right).
\end{aligned}
\tag{4.12}
$$

and

$$
\begin{aligned}
\sigma_{\vartheta\vartheta} = \sigma_{\varphi\varphi} &= \lambda \operatorname{tr}(\boldsymbol{e}) + 2\mu e_{\theta\theta} \\
&= \lambda\left[\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\left(1-\tfrac{1}{2}\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\right)+\tfrac{2u_r}{r}\left(1-\tfrac{1}{2}\tfrac{u_r}{r}\right)\right]+2\mu\tfrac{u_r}{r}\left(1-\tfrac{1}{2}\tfrac{u_r}{r}\right) \\
&= \lambda\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\left(1-\tfrac{1}{2}\tfrac{\mathrm{d}u_r}{\mathrm{d}r}\right)+2(\lambda+\mu)\tfrac{u_r}{r}\left(1-\tfrac{1}{2}\tfrac{u_r}{r}\right).
\end{aligned}
\tag{4.13}
$$

As for the equation to be solved, it also comes from the balance of linear momentum [12, § 3.2]:

$$
\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\rho\boldsymbol{f} \tag{4.14}
$$

After some simplifications, the one-dimensional form stays as

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r}+\frac{2}{r}\left[\sigma_{rr}-\sigma_{\vartheta\vartheta}\right]=-\rho f_r=+\rho\frac{Gm\left(r\right)}{r^2}\ . \tag{4.15}$$

The right hand side can be written also as follows:

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r}+\frac{2}{r}\left[\sigma_{rr}-\sigma_{\vartheta\vartheta}\right]=\rho\rho_0\frac{4\pi GR^3}{3r^2}=\frac{4\pi}{3}G\rho\rho_0\left(1-\frac{u_r}{r}\right)^3 r. \tag{4.16}$$

In the previous expression (4.16) there appear two different values referring to density. ρ_0 stands for the density in the referential configuration. The second one ρ is the density after deformation. Notice that both may vary in space.
The numerical difference between both densities is explained below:

$$\rho=\frac{\rho_0}{J}\ , \tag{4.17}$$

where J is the Jacobian of the deformation gradient.

$$J=\det\left(\boldsymbol{F}\right)=\frac{1}{\det\left(\boldsymbol{F}^{-1}\right)}\Rightarrow\frac{1}{J}=\det\left(\boldsymbol{F}^{-1}\right)=\frac{\mathrm{d}R}{\mathrm{d}r}\left(\frac{R}{r}\right)^2=\left(1-\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)\left(1-\frac{u_r}{r}\right)^2. \tag{4.18}$$

Therefore the right hand side of the equation evolves

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}x}+\frac{2}{x}\left[\sigma_{rr}-\sigma_{\vartheta\vartheta}\right]=\frac{4\pi}{3}G\rho_0^2\left(1-\frac{\mathrm{d}u_r}{\mathrm{d}r}\right)\left(1-\frac{u_r}{r}\right)^5 r=\frac{4\pi}{3}G\rho_{\mathrm{E}}^2 r_{\mathrm{E}}^2\left(1-u'\right)\left(1-\frac{u}{x}\right)^5 x\,. \tag{4.19}$$

Since we do not know the density of the Earth in the referential placement, we approximate it as the mean density of the Earth nowadays, calculated as in equation (4.1), even though the referential one is greater. Notice that in the last step it was introduced $x=\frac{r}{a}$ in which a is defined as the outer radius of the Earth in the referential configuration and it is approximated to the outer radius of the Earth nowadays. Then, also displacements changed from the ones described on the radial axis (u_r) to the ones described on the x-axis (u). The next expressions explain some changes already done or that will be done in the future, in order to normalize the radial axis:

$$r=xa,\quad u_r=ua,\quad \tfrac{\mathrm{d}u_r}{\mathrm{d}r}=\tfrac{a\,\mathrm{d}u}{a\mathrm{d}x}=u',\quad \tfrac{\mathrm{d}^2u_r}{\mathrm{d}r^2}=u''\ . \tag{4.20}$$

With stresses already described and the equation set, we are able to continue with the study. But in order to analyze the results easily, we normalize the expressions used to solve the equation. This normalization will be done, also in this case, with the Bulk Modulus k. The equation stays as follows:

$$\frac{\mathrm{d}\tilde{\sigma}_{rr}}{\mathrm{d}x}+\frac{2}{x}\left[\tilde{\sigma}_{rr}-\tilde{\sigma}_{\vartheta\vartheta}\right]=\frac{4\pi}{3k}G\rho_{\mathrm{E}}^2a^2\left(1-u'\right)\left(1-\frac{u}{x}\right)^5 x\,, \tag{4.21}$$

and the normalized stresses change to

$$\begin{aligned}\tilde{\sigma}_{rr} &= \frac{\sigma_{rr}}{k} = \frac{(\lambda+2\mu)}{k}u'\left(1-\frac{u'}{2}\right)+\frac{2\lambda}{k}\frac{u}{x}\left(1-\frac{1}{2}\frac{u}{x}\right)\\ &= \frac{3\,(1-\nu)}{1+\nu}u'\left(1-\frac{u'}{2}\right)+\frac{6\nu}{1+\nu}\frac{u}{x}\left(1-\frac{1}{2}\frac{u}{x}\right)\end{aligned} \tag{4.22}$$

and

$$\tilde{\sigma}_{\vartheta\vartheta} = \tilde{\sigma}_{\varphi\varphi} = \frac{\tilde{\sigma}_{\vartheta\vartheta}}{k} = \frac{\lambda}{k}u'\left(1-\frac{u'}{2}\right)+\frac{2\,(\lambda+\mu)}{k}\frac{u}{x}\left(1-\frac{1}{2}\frac{u}{x}\right) = \frac{3\nu}{1+\nu}u'\left(1-\frac{u'}{2}\right)+\frac{3}{1+\nu}\frac{u}{x}\left(1-\frac{1}{2}\frac{u}{x}\right). \tag{4.23}$$

Notice the difference between the expressions (4.12) and (4.13) to the previous ones (4.22) and (4.23). Not only the expression is divided by k but also the variable is changed from radial (r) to normalized distance (x) following the changes described in (4.7). Moreover, both first λ and second μ LAMÉ parameters are replaced with the equivalent expression using only the POISSON's Ratio ν in order to reduce the model from three (α, μ and λ) to two constants (α and ν).
The right hand side of the equation (4.21) can be shorten by inventing a parameter alpha (α) that depends on the material properties and dimension of the object under study. The expression is rewritten here:

$$\alpha = \frac{4\pi G\rho_{\mathrm{E}}^2 a^2}{3k}. \tag{4.24}$$

In this model had been used the next values for the constants:

ρ_{E}	The mean density of the Earth and equals to 5515 kg/m³.
E_0	For the YOUNG's modulus the value is the one of the steel: $E = 210 \cdot 10^9$ MPa.
ν	For Poisson's ratio the one used is also the equivalent in steel: 0.3
a	The current value of the outer radius of the Earth is equal to 6378137 m.

4.1.2 Boundary Conditions

First of all, the displacement $\boldsymbol{u}$ in the core has to be zero. The core in the final placement has to be in the same position as in the beginning of the study, as soon as there are no external forces applied to the system. This boundary condition is DIRICHLET (or first-type boundary condition) because is set directly on the solution [14] and can be written like this:

$$\boldsymbol{u}|_{x=0} = 0. \tag{4.25}$$

The second and last boundary condition is NEUMANN (also called second-type) because involves the derivative of the solution, among others [16]. In this case, we set that the stress in the direction of the radius r in the crust has to be equal to zero. It is like this because according to the third NEWTON's law of motion, when one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body

[18]. There is no force applied in the crust from outside, hence there cannot be any force applied from the inside. It can be written as follows:

$$\tilde{\sigma}_{rr}\big|_{x=1} = \left.\frac{\sigma_{rr}}{k}\right|_{x=1} = 0\,. \tag{4.26}$$

4.1.3 Weak form

At this point, we only need to write the equation (4.21) in a way that the software can solve. This means using the Variational Formulation. It will result into transforming the equation into the weak form. To carry out this purpose, we will need a test function that we will call δu made out of finite elements.
First of all, we multiply both sides by this test function δu and we integrate them along the whole domain (Ω). The domain in this case means that the x goes from 0 (core of the Earth) to 1 (crust of the sphere). It results in the following:

$$\int_0^1 \left(\tilde{\sigma}'_{rr} + \frac{2}{x}\left[\tilde{\sigma}_{rr} - \tilde{\sigma}_{\vartheta\vartheta}\right]\right)\delta u\,\mathrm{d}x = \int_0^1 \alpha\left(1-u'\right)\left(1-\frac{u}{x}\right)^5 x\,\delta u\,\mathrm{d}x\,, \tag{4.27}$$

and we can continue using integration by parts:

$$-\int_0^1 \tilde{\sigma}_{rr}\frac{\mathrm{d}\delta u}{\mathrm{d}x}\,\mathrm{d}x + \left[\tilde{\sigma}_{rr}\delta u\right]_0^1 + \int_0^1 \frac{2}{x}\left[\tilde{\sigma}_{rr} - \tilde{\sigma}_{\vartheta\vartheta}\right]\delta u\,\mathrm{d}x = \int_0^1 \alpha\left(1-u'\right)\left(1-\frac{u}{x}\right)^5 x\,\delta u\,\mathrm{d}x\,. \tag{4.28}$$

This is meant to be the weak form introduced in the code. Notice that the second term of the previous equation (4.28) equals to zero. Then, $\left[\tilde{\sigma}_{rr}\delta u\right]_0^1 = 0$ stands to the NEUMANN boundary condition written down in the weak form. Therefore

$$\tilde{\sigma}_{rr}\delta u\big|_{x=1} = 0\,, \tag{4.29}$$

and

$$\tilde{\sigma}_{rr}\delta u\big|_{x=0} = 0\,, \tag{4.30}$$

since the DIRICHLET boundary condition already sat

$$\delta u\,(x=0) = 0\,. \tag{4.31}$$

In order for the software to solve it, we will have to define function spaces, domains, facets and boundaries amongst others. Moreover, FEniCS needs to approximate the unknown function, displacement in this case u, into a trial function to be able to reach the solution. It means that we have to distinguish between trial and test space functions in the code.
the final weak form is

$$-\int_0^1 \tilde{\sigma}_{rr}\frac{\mathrm{d}\delta u}{\mathrm{d}x}\mathrm{d}x + \int_0^1 \frac{2}{x}\left[\tilde{\sigma}_{rr} - \tilde{\sigma}_{\vartheta\vartheta}\right]\delta u\,\mathrm{d}x = \int_0^1 \alpha\left(1-u'\right)\left(1-\frac{u}{x}\right)^5 x\,\delta u\,\mathrm{d}x\,. \tag{4.32}$$

4.1.4 Results

Analytical Solution First of all, the analytical solution is going to be exposed in order to have something to compare with the numerical results:

$$u_{\text{anal}}(x) = -\frac{\alpha_{\text{new}}}{10}\left(\frac{3-\nu}{1+\nu} - x^2\right)x\,, \tag{4.33}$$

$$\tilde{\sigma}_{rr,\,\text{anal}}(x) = -\frac{\alpha_{\text{new}}}{10}\,\frac{3\,(3-\nu)}{1+\nu}\left(1-x^2\right) \tag{4.34}$$

and

$$\tilde{\sigma}_{\vartheta\vartheta,\,\text{anal}}(x) = -\frac{\alpha_{\text{new}}}{10}\,\frac{3\,(3-\nu)}{1+\nu}\left(1-\frac{1+3\nu}{3-\nu}x^2\right)\,. \tag{4.35}$$

This new parameter (α_{new}) is taken from the beta-formulation exposed in the next section 4.2. It is calculated like this:

$$\alpha_{\text{new}} = \frac{4\pi G\rho_0^2 a^2}{3\,(\lambda+2\mu)}\,. \tag{4.36}$$

Numerical Solution There are two parameters in which the solution depends:

α parameter α is calculated as described in equation (4.24). It expresses a ratio between gravity and stiffness. It depends on material and geometrical parameters.

Resolution this parameter expresses the number of elements that the mesh has. As more elements, more precision of the results but it requires more time to achieve the solution. It refers in this paper to the maxim number of elements that the mesh can get before the model does not converge.

The solution found by finite elements method shows that the higher value of α to converge in the simulation is $\alpha_{\text{max}} = 1.1241$. Hence, this final value is not high enough yet to study the case of the Earth.
In Figure 4.1 there are plotted some distributions of displacement along the radial axis, depending on the different values of alpha. There is not only the numerical solution but also the analytical one, in order to have some values to compare with. At lower values of alpha, the solution has lots of resemblances with the u-formulation in the referential placement.
Notice that in the same plot (Figure 4.1), the maximum value of alpha plotted is not α_{max} but 1.3. This is because solutions taken with alphas higher than 1.1241 did not converge, but the values could still be plotted. In fact, the displacement when alpha is equal to 1.2 and 1.3 do not show the same shape as the previous plots. Nonetheless, the actual values plotted do not change much from the tendency followed by the convergent plots. Of course, as alpha increases, the plots start to curve and show totally different shapes and values. Therefore, as said before this model is not able to calculate the displacement for the Earth values.
We can notice the following from plot in Figure 4.1:

- All displacements are negative, as expected in a contraction.

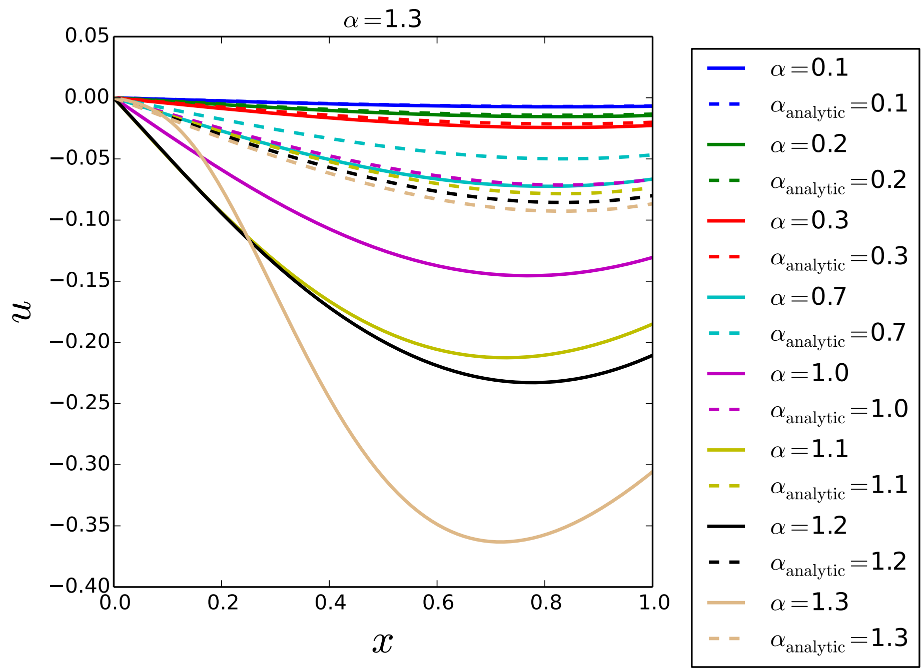


Figure 4.1: Plot of displacement for different values of α along the radial axis

- As α increases, the modulus of normalized displacement u increases.
- As α increases, the difference between the analytical and the numerical solution widens.
- For each α a point can be found in which the normalized displacement u is the minimum and it is not found in the crust ($x = 1$). This supports LOVE's theory 3.1.2.
- This LOVE radius is found around $x = 0.8$. It seems to decrease slightly as we increase the value of α.
- There is also a difference between the minimum radius of the analytical and the numerical solution (of same values of α).
- The slope near the center of the sphere ($x = 0$) increases as we increase α.

As for the resolution, the final value of alpha (α_{max}) can be calculated with a resolution of only 1000 divisions of the mesh. In the following graphic (Figure 4.2) can be appreciated the maximum values of resolution for each alpha. Notice that the tendency is that the resolution decreases when alpha increases. For instance, when alpha is 0.1 the resolution is 51630±10 and when alpha takes a value near the limit of convergence such as 1.1 the resolution decreases until 23430±10. If we increased this maximum resolutions, the numerical model would not converge any more. We assume that these non-convergence is due to numerical problems caused by the type of solver used, in our case a non-linear one.
In the appendix of this same paper, the code used to solve the differential equation with FEniCS [1] is attached.

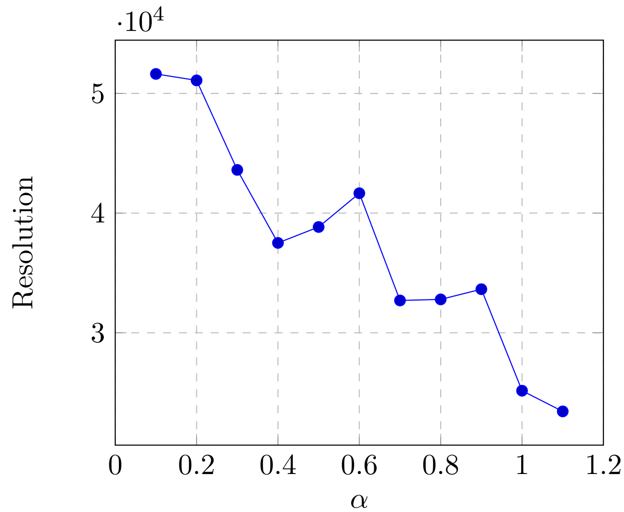


Figure 4.2: Maximum number of divisions of the mesh depending on value of alpha

4.1.5 Abstract

With this study we found some useful information according to how the solution changes depending on alphas and difference between analytical and numerical solution. But be aware that with this method no solution is found for the case of the Earth ($\alpha_{\text{Earth}} = 1.97629$). The current alpha limit is only around 1.1241.

The current model have been solved with a non-linear solver, because the equation to solve is non linear. Therefore, it takes more time to solve it and we found that not always the solution converges, being this a limit for our model.

To try to widen the interval of alphas (α) that our model can solve, we can change the displacement u, the unknown variable in our model, into another one called beta ($\beta = 1 - \frac{u}{x}$). This change makes the weak form easier to be solved by the computer, since both expressions of the stresses get simplified and more easily operated by the solver.

4.2 Beta Formulation

4.2.1 Variables and Equations

All equations in this formulation are strictly the same as the equation in the previous formulation (4.1). Nonetheless, the aspect of these equations and expressions change because instead of solving the system with displacement as the unknown function we solve another function called β.

We are introducing the next change

$$\beta = 1 - \frac{u}{x}\,, \tag{4.37}$$

this change also implies the following changes:

$$u_r = r\,(1-\beta) \Rightarrow u_r' = 1 - \beta - r\beta', \tag{4.38}$$

and

$$u = x\,(1-\beta) \Rightarrow u' = 1 - \beta - x\beta'\,. \tag{4.39}$$

Replacing all displacements u and displacements derivatives u' in the previous equations (4.21), (4.22) and (4.23), we obtain:

$$\tilde{\sigma}_{rr} = -\frac{3\lambda + 2\mu}{2k}\left(1-\beta^2\right) - \frac{\lambda+2\mu}{2k} r^2 \left(\beta'^2 + \frac{2\beta\beta'}{r}\right), \tag{4.40}$$

$$\tilde{\sigma}_{\vartheta\vartheta} = \tilde{\sigma}_{\varphi\varphi} = -\frac{\lambda}{2k} r^2 \left(\beta'^2 - \frac{2\beta\beta'}{r}\right) - \frac{3\lambda+2\mu}{2k} r^2 \left(1-\beta^2\right) \tag{4.41}$$

and the equation with which is going to be taken out the weak form looks like this:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[3\left(\lambda+2\mu\right)\beta^2 + \left(\lambda+2\mu\right)\left(\beta'^2 + \frac{2\beta\beta'}{r}\right) r^2\right] + 4\mu r \beta'^2 = 2\rho_{\mathrm{E}} f_r\,. \tag{4.42}$$

The right hand side of the equation can evolve applying the Newton's law of universal gravitation [17] as follows:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[3\left(\lambda+2\mu\right)\beta^2 + \left(\lambda+2\mu\right)\left(\beta'^2 + \frac{2\beta\beta'}{r}\right) r^2\right] + 4\mu r \beta'^2 = -\frac{8G\pi\rho_{\mathrm{E}}^2 a^2}{3\left(\lambda+2\mu\right)}\left(\beta + r\beta'\right)\beta^5\,. \tag{4.43}$$

4.2.2 Boundary Conditions

The boundary condition in this case is set on β since it is the unknown function now. But in fact, the boundary condition is set in the time derivative of β in the center of the sphere ($x = 0$). It is in the center of the sphere that it has to be imposed that the slope of beta is equal to 0. Otherwise, β (and consequently the displacement) would have a non-derivable point in the center of the Earth. This condition is written down as follows:

$$\beta'\big|_{x=0} = 0\,. \tag{4.44}$$

4.2.3 Weak form

After multiplying by the test function $\delta\beta$ and integrating both sides of the equation that comes out from the balance of linear momentum (4.14) and also applying Gauss theorem, the final weak form results in:

$$-\int_0^1 \tilde{\sigma}_{rr}\frac{\mathrm{d}\delta\beta}{\mathrm{d}x}\mathrm{d}x - \frac{3}{2}\int_0^1 \left(1-\beta^2\right)\delta\beta \mathrm{d}S + \int_0^1 \frac{2}{x}\left(\tilde{\sigma}_{rr} - \tilde{\sigma}_{\vartheta\vartheta}\right)\delta\beta\mathrm{d}x + \alpha\int_0^1 \left(\beta' x + \beta\right) x\beta^5\delta\beta\mathrm{d}x = 0\,. \tag{4.45}$$

4.2.4 Results

In this case, the solution is analyzed directly with β. But in order to do it correctly and have some analytical solution to compare with, Professor W.H. Müller delivered the expression of $\beta_{\mathrm{anal}}(x)$, which is the following:

$$\beta_{\mathrm{anal}}(x) = \frac{\alpha_{\mathrm{M}}}{20}\left(\frac{3-\nu}{1+\nu} - x^2\right) + 1\,. \tag{4.46}$$

In the previous expression (4.46) a new parameter is included. This α_{M} is referred as "alpha MÜLLER" and is calculated as follows:

$$\alpha_{\mathrm{M}} = \frac{8\pi G\rho_0^2 r^2}{3\,(\lambda + 2\mu)} = \frac{4\pi G\rho_{\mathrm{E}}^2 r^2}{3k}\frac{2k}{\lambda + 2\mu} = \alpha\frac{2}{3}\frac{1+\nu}{1-\nu}\,. \tag{4.47}$$

In the expression above (4.47), there is shown the relation between the previous used alpha (α) described as (4.24) and the new alpha (α_{M}). Merging the previous two equations we obtain the final expression of $\beta_{\mathrm{anal}}\,(x)$ depending in the usual alpha:

$$\beta_{\mathrm{anal}}\,(x) = \frac{\alpha}{30}\frac{1+\nu}{1-\nu}\left(\frac{3-\nu}{1+\nu} - x^2\right) + 1\,. \tag{4.48}$$

With it and the numerical solution of beta, we can take out the following two plots (Figure 4.3) for different values of alpha in which both solutions can be compared.

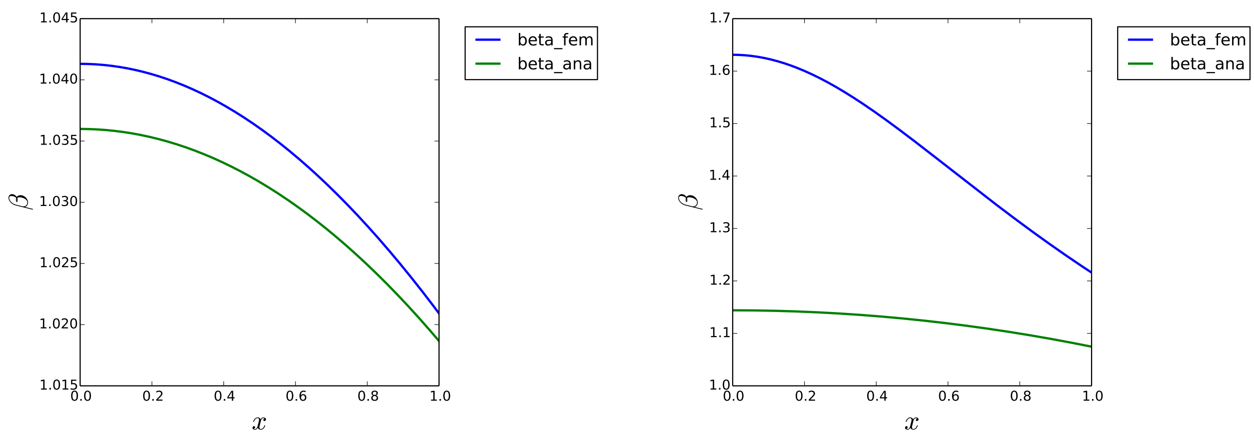


Figure 4.3: Comparative plots of β_{num} and β_{anal} for two different alphas

In the first plot of Figure 4.3, it can be seen the solution for $\alpha = 0.34655$. This value is equivalent to $\alpha_{\mathrm{M}} = 0.27991$ which is the alpha value for planet Mercury. It is represented this value, because Professor MÜLLER had some solutions with this concrete value for alpha in order to compare with. The difference between beta analytical and beta numerical is almost the same as the one he obtained, which let us think that this is the correct solution. In the initial value of beta (when x=0), the error between the numerical and the analytical one is equal to:

$$\delta|_{x=0,\,\alpha=0.34655} = \frac{1.0418 - 1.0362}{1.0418}100 = 0.53\%\,. \tag{4.49}$$

This difference between the numerical and the analytical solution δ (4.49) is not even of the one percent, therefore we consider the solution good enough. Moreover, in both plots the slope of beta in the center of the sphere ($x = 0$) is clearly 0, which makes us notice that the boundary conditions worked as planned.

It cannot be said the same when the value of alphas increase. Near to the limit value for alpha for this case, the difference δ increases considerably. For instance, when alpha equals to 1.12 (4.3) δ increases until:

$$\delta|_{x=0,\,\alpha=1,12} = \frac{1.636 - 1.149}{1.636} 100 = 29.8\% \,. \tag{4.50}$$

Hence, this beta formulation for the current location is not yet a good approximation for high values of alpha.
As for the range of alphas which the code is able to solve, even when the code is not exactly the same as in the previous section (4.1), the maximum value of alpha is still the same as before ($\alpha_{\max} = 1.1241$). However, the resolution for the mesh is not equal as in the displacement formulation. It is much lower as Figure 4.4 shows. Furthermore, the tendency followed by the points in the graph is not clear this time, and not even descendant.

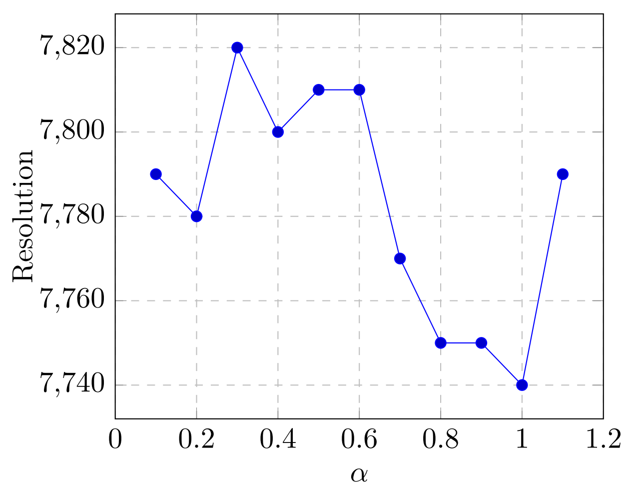


Figure 4.4: Maximum resolutions for the mesh depending on alpha

In the Appendix A.2 can be seen the whole code used to solve the problem under the current configuration in the beta-formulation. Moreover, it has some extra lines which are the ones in charge to plot the solution.

4.2.5 Abstract

When the problem is solved with a model based on the function β, the solution obtained is not yet satisfactory for alpha values close the the Earth. In fact, when the alpha value increases almost until the limit ($\alpha = 1.12$), the difference between the analytical and the numerical solution escalates until the thirty percent. Therefore, the result is not accurate and the maximum alpha value is not close to the Earth one.
Thus, this beta-formulation in the current placement does not achieve the goal of being capable to solve numerically the case of the Earth.

4.3 Summary

In the present section the same problem has been solved using the same equations and expressions, but with different notations. In both formulations the last solution which converged was under

the same value of $\alpha_{\max} = 1.1241$. Nonetheless, the resolutions of the mesh were more accurate in the case of the displacement formulation.
The difference between the analytical solution and the numerical one increased when increasing the value of alpha. Therefore, non of the two formulations are useful for the case of the Earth.

5 Rheological Studies

5.1 Introduction

In this subsection all studies consider a linear visco-elastic model. All previous subsections calculated stresses and displacements under the hypothesis that those only depended on strains. Otherwise, it is going to be considered from now on that they also depend on the past history of the strains suffered by the body. When it occurs it results on some delay in the internal adjustments of the body [7, § 12.4.1]. Moreover, taking into account that the study will be done without any second order derivative it takes us to a linear study.
Starting with Gibbs' equation and carrying a thermodynamical study described in [7, §12.4.2] we get to the next equation for one-dimensional models:

$$\tau_\sigma \frac{\mathrm{d}\sigma}{\mathrm{d}t} + \sigma = E_0 \left(\tau_\varepsilon \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \varepsilon \right) . \tag{5.1}$$

The previous equation (5.1) describes the (σ, ε)-relation of linear visco-elastic materials. It is called Rheological equation of state. τ_σ and τ_ε are known as relaxation times and E_0 is the static elastic modulus, considered to be appropriate in very small changes of σ and ε.
In the following cases, the dimensions of the beam are $x = 10$, $y = 1$ and $z = 1$ all of them in meters. Therefore, the beam has a square section with a length 10 times more than the side of the section.

5.2 Case 1: 1D Beam under constant external stress

5.2.1 Introduction and Hypothesis

This part of the thesis studies a simplified one-dimensional beam under a constant external stress. This beam is supposed to be build in a wall on one end and on the other end the constant stress is applied. Moreover, in this first case, the mass of the beam is neglected. Figure 5.1 illustrates these conditions.
It is going to be studied how the beam responds to a sudden stress described as follows:

$$\sigma(t) = \begin{cases} 0 & t \leq 0 \\ \sigma_0^{\text{ext}} & t > 0 \end{cases} . \tag{5.2}$$

It means that before the study begins, the beam is resting and the strain is also considered equal to zero ($\varepsilon(t) = 0$ for $t \leq 0$), but as the time starts to run, the stress is constant and with a value equal to σ_0^{ext}. Therefore, the strain is going to be the solution to be analyzed.
Moreover, the process is meant to be isothermal to avoid members depending on thermal conditions.

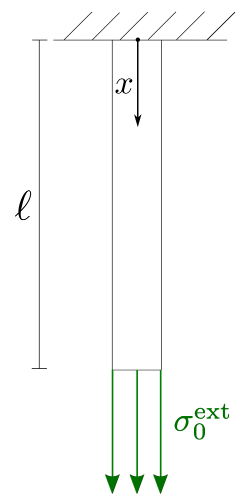


Figure 5.1: Sketch representing the system under study

5.2.2 Analytical solution

To reach the analytical solution we consider the balance of linear momentum (4.14) under the hypothesis taken in the previous subsection (5.2.1). It stays as follows:

$$\frac{\partial \sigma}{\partial x} = 0\,. \tag{5.3}$$

From this simple equation we get that the stress (σ) has to be constant (as we already knew). Moreover, the equation of state is also needed. In this case, the Rheological one (5.1) has to be solved. It means finding the expression of strain depending on time $\varepsilon\,(t)$. That is one of the simplest differential equations because the derivatives are all derived by the same variable (time, t). First of all we depart from an "ansatz", considering that the stress is a constant:

$$\varepsilon_{\mathrm{hom}}\,(t) = \mathrm{C}\mathrm{e}^{\lambda \mathrm{t}}\,, \tag{5.4}$$

and also its derivative

$$\frac{\mathrm{d}\varepsilon_{\mathrm{hom}}}{\mathrm{d}t} = \mathrm{C}\lambda\mathrm{e}^{\lambda \mathrm{t}}\,. \tag{5.5}$$

These two expressions (5.4 and 5.5) can be substituted in the Rheological equation of state (5.1) bearing in mind that the derivative of stress in space is 0 (5.3), the result is the homogeneous equation:

$$E_0\left(\tau_\varepsilon \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + \varepsilon\right) = 0\,. \tag{5.6}$$

With this information we can get the value of λ. Therefore, the homogeneous solution to the differential equation is:

$$\varepsilon_{\mathrm{hom}} = \mathrm{C}\,\mathrm{e}^{-\frac{t}{\tau_\varepsilon}}\,. \tag{5.7}$$

To get the particular equation the "ansatz" is the same as the previous but the constant (C) now depends on time. Therefore:

$$\varepsilon_{\text{part}}\,(t) = \mathrm{C}\,(t)\,\mathrm{e}^{-\frac{\mathrm{t}}{\tau_\varepsilon}}\,, \tag{5.8}$$

and

$$\frac{\mathrm{d}\varepsilon_{\text{part}}}{\mathrm{d}t} = \frac{\mathrm{dC}}{\mathrm{d}t}\mathrm{e}^{-\frac{\mathrm{t}}{\tau_\varepsilon}} + \mathrm{C}\,(t)\left(-\frac{1}{\tau_\varepsilon}\right)\mathrm{e}^{-\frac{\mathrm{t}}{\tau_\varepsilon}}\,. \tag{5.9}$$

When substituting this two expressions (5.8) and (5.9) in the complete equation of state (5.1) and integrating in both sides apart from using other mathematical resources, the final particular solution has the following appearance:

$$\varepsilon_{\text{part}} = \frac{1}{E_0}\frac{\tau_\sigma}{\tau_\varepsilon}\left[\sigma_0^{\text{ext}}\right] - \frac{1}{E_0}\frac{\tau_\sigma}{\tau_\varepsilon}\sigma_0^{\text{ext}}\left[1-\mathrm{e}^{-\frac{\mathrm{t}}{\tau_\varepsilon}}\right] + \frac{\sigma_0^{\text{ext}}}{E_0}\left[1-\mathrm{e}^{-\frac{\mathrm{t}}{\tau_\varepsilon}}\right]\,. \tag{5.10}$$

Moreover, on the bibliography I. Müller and W.H. Müller [7, §12.4.3] also reached the same solution as the particular one (5.10) but already simplified

$$\varepsilon = \frac{1}{E_0}\frac{\tau_\sigma}{\tau_\varepsilon}\left[\sigma_0^{\text{ext}}\right] + \frac{1}{E_0}\left(1-\frac{\tau_\sigma}{\tau_\varepsilon}\right)\sigma_0^{\text{ext}}\left[1-\mathrm{e}^{-\frac{\mathrm{t}}{\tau_\varepsilon}}\right]\,, \tag{5.11}$$

which confirms that the calculi above are well done.
The next step consists in making a time normalization to the previous equation in order to know at what time will the solution reach the stationary point. The time normalization consists in applying the next change into the model:

$$t = \tau_\varepsilon \tilde{t} \tag{5.12}$$

and therefore

$$\mathrm{d}t = \tau_\varepsilon \mathrm{d}\tilde{t}\,. \tag{5.13}$$

In the previous two expressions (5.12) and (5.13) the variable $\tilde{t}$ represents the normalized time. Once applied the change, the analytical solution shows the next appearance:

$$\varepsilon = \frac{1}{E_0}\frac{\tau_\sigma}{\tau_\varepsilon}\left[\sigma_0^{\text{ext}}\right] + \frac{1}{E_0}\left(1-\frac{\tau_\sigma}{\tau_\varepsilon}\right)\sigma_0^{\text{ext}}\left[1-\mathrm{e}^{-\tilde{\mathrm{t}}}\right]\,, \tag{5.14}$$

With this simple procedure, which almost does not change the equation, we can know before calculating anything, at what time ($\tilde{t}$) the solution is going to reach the stationary point. It is like this, because every equation which has a variable inside an exponential like this (e^x) will reach approximately the 98% of the final value in 4 times x [5]. In our case, the variable x is the normalized time ($\tilde{t}$) so the end time in the code will be considered as $\tilde{t} = 4$.
With the next random values of the variables:

- $\sigma_0^{\text{ext}} = 10^8$ MPa
- $E_0 = 210 \cdot 10^9$ MPa
- $\tau_\varepsilon = 0.05$
- $\tau_\sigma = 0.01$

The analytical solution is the plotted in Figure 5.2. In it we can appreciate that the final value of the strain (ε) is around 0.00047. With it, we can also prove that the time normalization worked as expected, because it can be seen the whole evolution of the solution.
Furthermore, the strain does not start in $\varepsilon = 0$ (when $t = 0$), but it starts with a strain equal to

$$\varepsilon|_{t=0} = \frac{\sigma_0^{\text{ext}}}{E_0} \frac{\tau_\sigma}{\tau_\varepsilon} = \frac{10^8}{210 \cdot 10^9} \frac{0.01}{0.05} = 9.524 \cdot 10^{-5} \,, \tag{5.15}$$

as expected if we evaluate the equation (5.11) when $t = 0$.

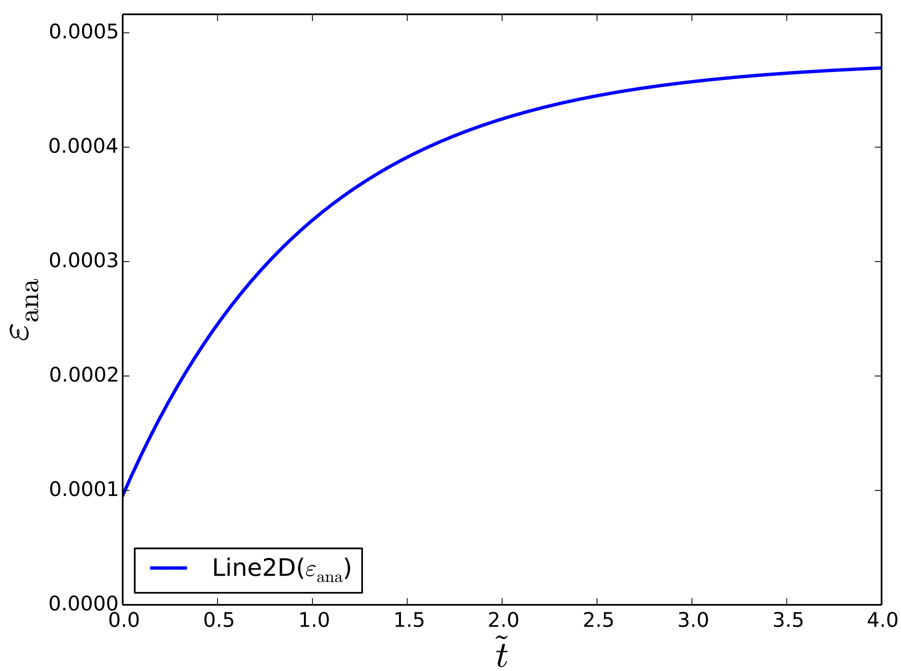


Figure 5.2: Evolution of the strain depending on normalized time

5.2.3 Numerical solution

Weak form In order to find the weak form to make FEniCS able to solve the equation, we depart from the balance of linear momentum (4.14) and write it down according to the hypothesis taken. Therefore, the equation results as the expression (5.3) used before.
Introducing some changes in the material law (5.1) we manage to get the expression of the stress σ as follows, after introducing finite differences in it:

$$\frac{\tau_\sigma}{\Delta t} \left(\sigma - \sigma^0\right) + \sigma = E_0 \left(\frac{\tau_\varepsilon}{\Delta t} \left(\varepsilon - \varepsilon^0\right) + \varepsilon\right) \,, \tag{5.16}$$

where σ is the stress value on the current time step, σ_0 is the stress in the previous one and Δt is the time step.
From this equation it is possible to keep σ apart and assuming τ_ε and τ_σ are constant and homogeneous:

$$\sigma = E_0 \frac{\tau_\varepsilon + \Delta t}{\tau_\sigma + \Delta t}\varepsilon - E_0 \frac{\tau_\varepsilon}{\tau_\sigma + \Delta t}\varepsilon^0 + \frac{\tau_\sigma}{\tau_\sigma + \Delta t}\sigma^0 \,. \tag{5.17}$$

Departing from the Rheological material law described before (5.16), integrating both sides along the length of the beam and multiplying by a test function ($\delta\psi$) we reach the next expression:

$$-\int_0^\ell \sigma \frac{\delta\psi}{\mathrm{d}x}\mathrm{d}x + [\sigma\delta\psi]_0^1 = 0 \,. \tag{5.18}$$

Introducing the previous expression of stress (5.17) into the weak form that we have at the moment we reach the weak form:

$$-E_0 \frac{\tau_\varepsilon + \Delta t}{\tau_\sigma + \Delta t}\int_0^\ell \varepsilon \frac{\mathrm{d}\delta\psi}{\mathrm{d}x}\mathrm{d}x + E_0 \frac{\tau_\varepsilon}{\tau_\sigma + \Delta t}\int_0^\ell \varepsilon^0 \frac{\mathrm{d}\delta\psi}{\mathrm{d}x}\mathrm{d}x - \frac{\tau_\sigma}{\Delta t}\int_0^\ell \sigma^0 \frac{\mathrm{d}\delta\psi}{\mathrm{d}x}\mathrm{d}x + \sigma_0^{\mathrm{ext}}\ \delta\psi|_{x=\ell} = 0 \,. \tag{5.19}$$

Introducing normalization of stress, length and time described as follows:

$$\begin{aligned} \sigma &= E_0\tilde{\sigma} & & \\ x &= \ell\,\tilde{x} \Rightarrow & \mathrm{d}x &= \ell\,\mathrm{d}\tilde{x} \\ u &= \ell\,\tilde{u} \Rightarrow & \mathrm{d}u &= \ell\,\mathrm{d}\tilde{u} \\ t &= \tau_\varepsilon\tilde{t} \Rightarrow & \Delta t &= \tau_\varepsilon\Delta\tilde{t} \end{aligned} \,, \tag{5.20}$$

and also considering that

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} \,, \tag{5.21}$$

we get the final weak form that is going to be used in the code:

$$-\frac{\tau_\varepsilon\left(1+\Delta\tilde{t}\right)}{\tau_\sigma + \tau_\varepsilon\Delta\tilde{t}}\int_0^1 \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{x}}\frac{\mathrm{d}\delta\tilde{\psi}}{\mathrm{d}\tilde{x}}\mathrm{d}\tilde{x} + \frac{\tau_\varepsilon}{\tau_\sigma + \tau_\varepsilon\Delta\tilde{t}}\int_0^1 \frac{\mathrm{d}\tilde{u}^0}{\mathrm{d}\tilde{x}}\frac{\mathrm{d}\delta\tilde{\psi}}{\mathrm{d}\tilde{x}}\mathrm{d}\tilde{x} - \frac{\tau_\sigma}{\tau_\varepsilon\Delta\tilde{t}}\int_0^1 \tilde{\sigma}^0\frac{\mathrm{d}\delta\tilde{\psi}}{\mathrm{d}\tilde{x}}\mathrm{d}\tilde{x} + \tilde{\sigma}_0^{\mathrm{ext}}\ \delta\tilde{\psi}\Big|_{\tilde{x}=1} = 0 \,. \tag{5.22}$$

Boundary Conditions As for the boundary condition, the beam is fixed from one end, which means that the displacements in this end have to be 0. The expression representing this DIRICHLET condition introduced in the code is the following:

$$u(x=0) = 0 \,. \tag{5.23}$$

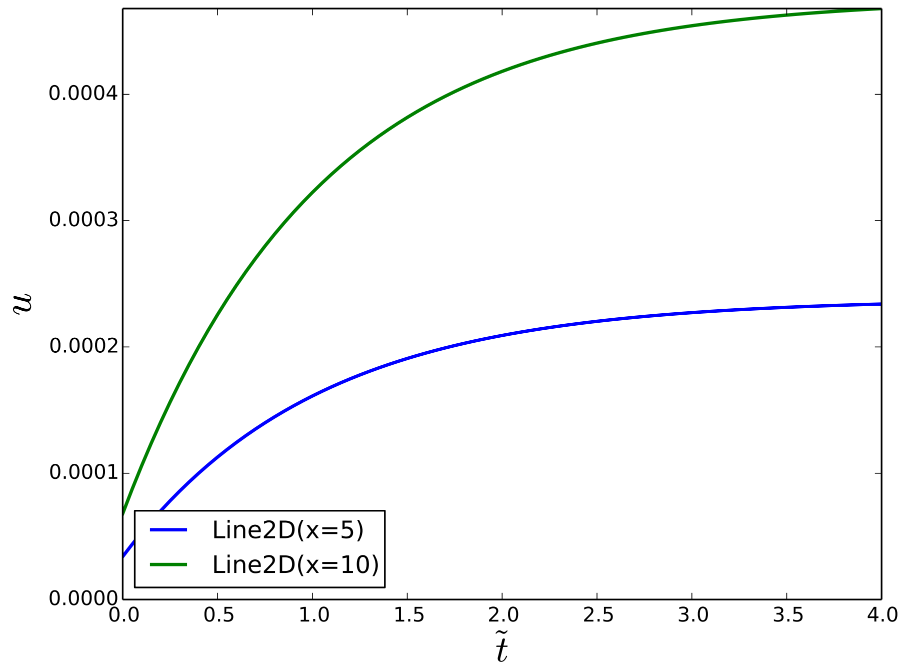


Figure 5.3: Displacement plots along time in two different positions of the beam

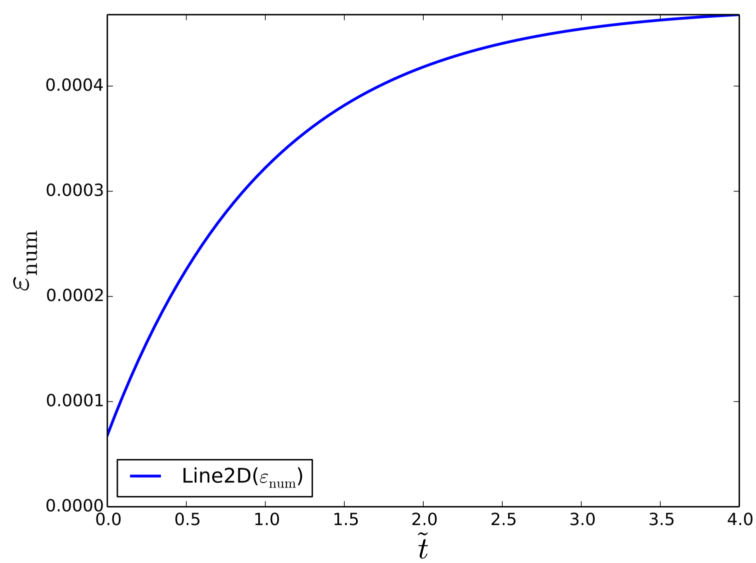


Figure 5.4: Evolution of the numerical strain depending on normalized time

Solution When analyzing the solution, we can plot both displacements and strains. On the one hand, we have figure 5.3, in which the evolution of displacements can be seen. In it there are plotted only the displacements of the half point of the beam and free end, on which the external force is applied.

Until this moment, there is nothing that makes us think that the results are not coherent. In the analytical solution there is no plot related to displacements. Therefore, in order to be able to compare results, we plot the numerical strain along the beam, depending on time (Figure 5.4) In this graphic (Figure 5.4) we can see that the strain at time 0 is the one according to the analytical solution in subsection 5.2.2. The value of the initial analytical strain is the one in equation (5.15) and compared with the numerical one

$$\varepsilon|_{\tilde{t}=0,\,\mathrm{num}} = 8.746 \cdot 10^{-5}\,, \tag{5.24}$$

we see that they are virtually the same.
As for the final value, the analytical one is

$$\varepsilon|_{\tilde{t}=4,\,\mathrm{anal}} = \frac{\sigma_0^{\mathrm{ext}}}{E_0} = \frac{10^8\,\mathrm{MPa}}{210 \cdot 10^9\,\mathrm{MPa}} = 4.762 \cdot 10^{-4}\,, \tag{5.25}$$

whereas the numerical is

$$\varepsilon|_{\tilde{t}=4,\,\mathrm{num}} = 4.680 \cdot 10^{-4}\,. \tag{5.26}$$

They can be considered almost the same.
Moreover, the random values of the time constants (τ_ε and τ_σ) are not that random. These where chosen because for higher values of the taus, the numerical solution would not converge. Nonetheless, since we already normalized the time, the shape and final value of the strain plot does not depend on these two values.

5.2.4 Abstract

This first case of a beam studied following the viscoelastic model is useful in order to get to know how strains, stresses and displacements can depend also on the time. Therefore, it is not enough to know the current placement of the system, but also the past history of it.
As for the analytical solution reached, it is the same one which is equal to the one on the bibliography. Moreover, it coincides totally with the numerical solution converged. Consequently, we reached a satisfying numerical solution.

5.3 Case 2: Constant stress and gravitational force

5.3.1 Introduction and Hypothesis

In this second case, the hypothesis are the ones described in the subsection 5.2.1, but adding the acceleration produced by the gravity g. This acceleration takes a mean value of 9.80665 m/s^2. As a consequence, the mass will be taking into consideration in this approach. The beam has a density of 8000 kg/m^3, which is the one of the steel. As for the dimensions of the beam are the same showed in section 5.1. As far as the values of other constants is concerned, they take the same values as the previous case (5.2). For a better comprehension of the system under study, the Figure 5.5 tries to solve all possible doubts.

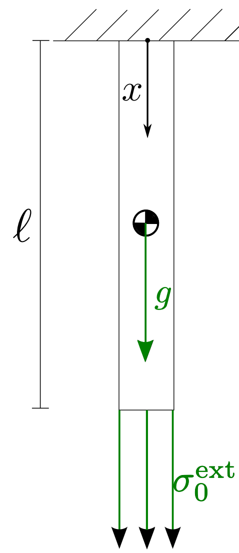


Figure 5.5: Sketch describing the system under study

5.3.2 Numerical solution

Weak form In order to find the equation to be solved with the help of the software, the procedure followed is the same as in paragraph 5.2.3. First of all, from the balance of linear momentum (4.14) we take out a first equation to be integrated:

$$\int_0^\ell \frac{\mathrm{d}\sigma}{\mathrm{d}x}\psi \mathrm{d}x = -\int_0^\ell \rho g \psi \mathrm{d}x \,. \tag{5.27}$$

In the previous equation (5.27) the variable ψ stands for the test function. Now applying GAUSS divergence theorem and the stress normalization by YOUNG's modulus ($\tilde{\sigma} = \frac{\sigma}{E_0}$):

$$-\int_0^\ell \tilde{\sigma} E_0 \frac{\mathrm{d}\psi}{\mathrm{d}x}\mathrm{d}x = -\int_0^\ell \rho g \psi \mathrm{d}x - \left[\tilde{\sigma} E_0 \psi\right]_0^\ell \,. \tag{5.28}$$

The resulting equation (5.28) is already the final weak form, which is introduced in the code. Nonetheless, this equation needs the expression of the normalized stress to be useful. From the rheological equation of visco-elastic materials (5.1), applying finite differences and also time normalization, we reach the final form of the stress:

$$\tilde{\sigma} = \frac{\sigma}{E_0} = \frac{\tau_\sigma \tilde{\sigma}^0 - \tau_\varepsilon \varepsilon^0}{\tau_\sigma - \tau_\varepsilon \Delta\tilde{t}} + \frac{\tau_\varepsilon \left(1 + \Delta\tilde{t}\right)}{\tau_\sigma + \tau_\varepsilon \Delta\tilde{t}}\varepsilon \,. \tag{5.29}$$

With this expression that describes the stress (5.29) and the weak form (5.28) the software needs only the boundary conditions to start the simulation.

Boundary Conditions In this second case, the beam is also fixed from the end, which means that the boundary conditions are exactly the same as in case 1 (5.2.3).

Solution

1st approach when $g = 0$ In order to compare the solution of case 2 with the solution in case 1, we solve the present case taking into consideration that $g = 0$. Figure 5.6 shows the strain calculated numerically when the gravitation is zero.

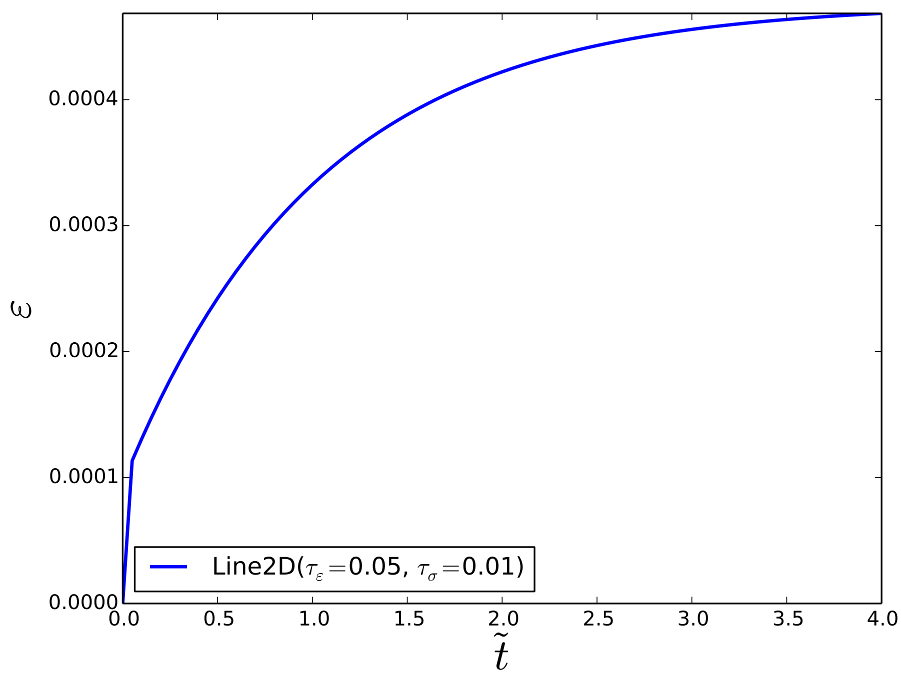
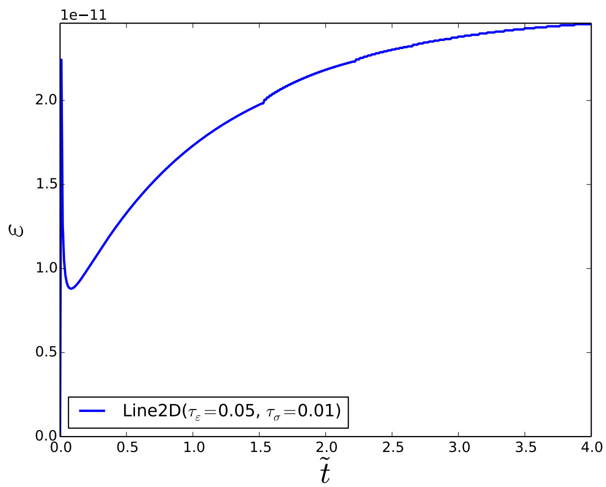


Figure 5.6: Strain numerically calculated when gravitation does not take part in the study

Comparing the plot of the analytical solution of case 1 (5.2) and the plot taken from the numerical solution of case 2 under the hypothesis $g = 0$ (5.6), we see that the final value is the same (ε =0.00047). Nonetheless, the numerical solution show that at the beginning of the simulation, the strain is 0, when it is supposed to be $9.524 \cdot 10^{-5}$ (expression 5.15). Therefore, the solution is not yet satisfying.

2nd approach when $\sigma_0^{\text{ext}} = 0$ If the numerical solution is calculated under the hypothesis of $\sigma_0^{\text{ext}} = 0$, the solution does not converge as easily as before. It is needed a smaller time step in order to achieve convergence. After different attempts, the strain calculated numerically shows the plot in Figure 5.7. This solution shown converged, but it does not follow an exponential shape in very first iterations. Therefore, there is a problem with the initial values probably caused by the numerical solver.
Moreover, in Figure 5.7 can be appreciated that the distribution along the length axis of the beam is not linear any more. This fact fits perfectly bearing in mind that in the model is only applied the stress caused by gravitation.

Complete solution When considered both the gravitation and the external pressure the solution shows a shape like the one in Figure 5.8. It shows a solution almost identical as the



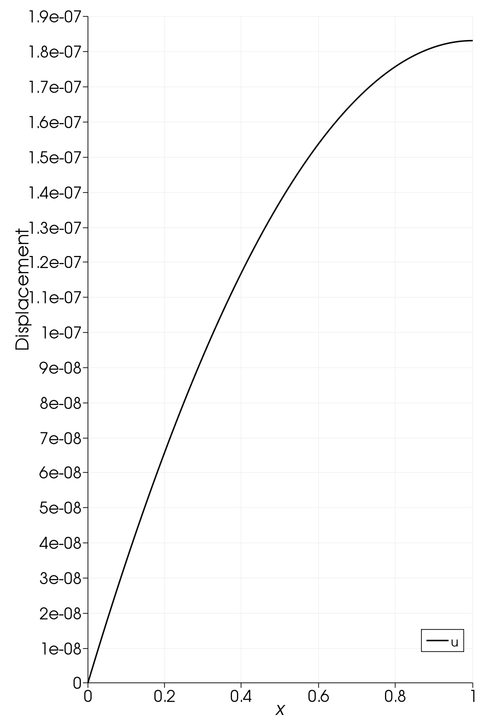


Figure 5.7: Strain and displacement according to the numerical solver when $\sigma_0^{\text{ext}} = 0$

1st approach when $g = 0$ (Figure 5.6), because the stress that the gravitation does ($\rho g x = 8000 \cdot 9.81 \cdot 10 = 7.85 \cdot 10^5$ MPa) is almost nothing compared to the external stress applied in the end of the beam (σ_0^{ext} =10^8 MPa). Therefore, in this plot the final solution of the strain should be coherent, but the initial one is not correct, since it cannot be 0.

5.3.3 Abstract

In this case we do not have any analytical solution to compare with the numerical one. Therefore there is no certainty that the solution is correct.
The numerical solution while the external stress is not considered ($\sigma_0^{\text{ext}} = 0$) converges, but shows some problem in the initial values. Nonetheless, after the first time steps, the plot follows an exponential function.
As far as the complete solution is concerned, we are not able to compare with any analytical solution. Therefore we have to wait until the next subsection (5.4) in which a similar case is solved. Nevertheless, the shape and values follow a coherent exponential equation.

5.4 Case 3: 3D Beam with gravitational and external stresses

5.4.1 Introduction and Hypothesis

The next step is not only considering a one-dimension model studying the length axis of the beam, but also how the transversal section respond to the stresses. Therefore, the present study is going to treat a 3D beam built-in a wall (zero degrees of freedom) on one end. On the other end there is applied a constant external stress. Moreover, the beam has a mass and, consequently, there is also a volumetric force caused by the gravity (Figure 5.9).

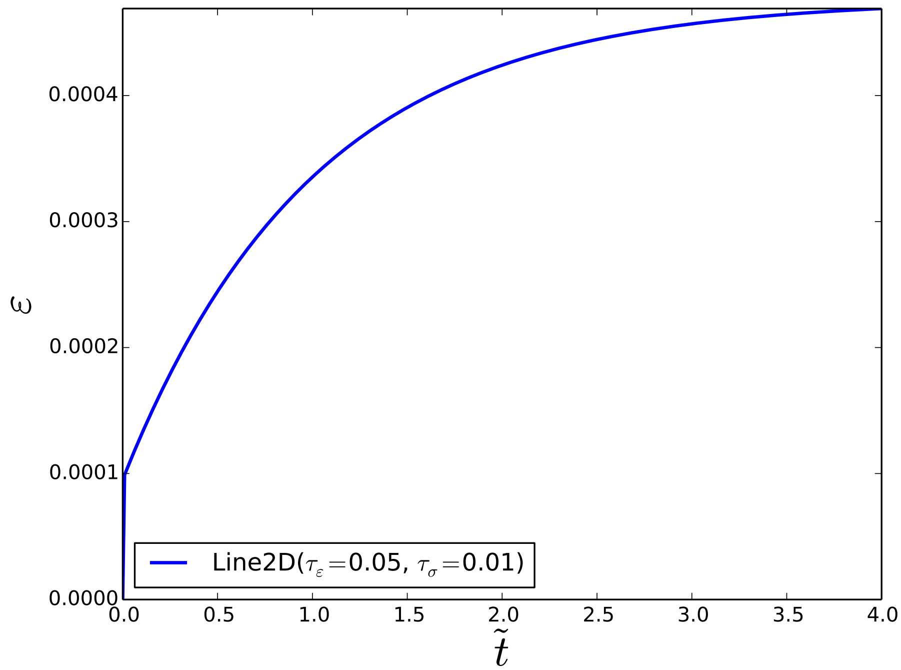


Figure 5.8: Complete strain depending on the time calculated numerically

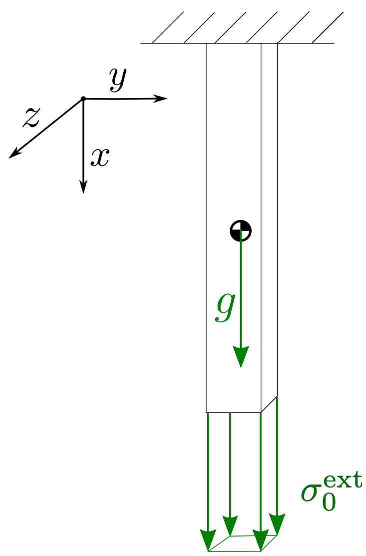


Figure 5.9: Sketch representing the stresses and acceleration present in the beam

It is used the Rheological equation of state also in this case as the material law that the beam is supposed to obey. As commented before (subsection 5.1), the model will depend now not only on the three space dimensions but also on the time.

5.4.2 Variables and Equations

The base of the study is, once more, the balance of linear momentum (4.14). In this very case, we can rewrite the equation as follows:

$$\frac{\partial \sigma_{ij}}{\partial x_i} = -\rho g_j \,, \tag{5.30}$$

where the gravitation is now a vector

$$g_j = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} , \tag{5.31}$$

and

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix} . \tag{5.32}$$

The constant g stands for the gravitation constant in the Earth's crust (9.80665 m/s^2). Furthermore the stress tensor is symmetric and it is written in N/mm^2 or MPa, which is equivalent.
The second equation needed to proceed is the one describing the visco-elastic behavior of the material. It can be found previously for one-dimensional cases as equation (5.1). In order to make it suitable for three-dimensional models there are some changes that have to be done.
First of all, we are going to split the stress tensor in two parts in order to make the program able to solve derivatives in 3 dimensions. This two parts are called "volumetric" and "deviatoric" parts. The volumetric part is the one in charge to describe how the volume of the body changed and the deviatoric gives information about the deformation suffered by the body [2, §1.2]. The trace of the deviatoric part is equal to zero since it does not change the volume of the body, it only deforms it. There are more details below:

$$\sigma_{ij} = \sigma_{ij}^{\text{vol}} + \sigma_{ij}^{\text{dev}} \,, \tag{5.33}$$

where

$$\sigma_{ij}^{\text{vol}} = \frac{1}{3}\sigma_{kk}\delta_{ij} \,, \tag{5.34}$$

hence

$$\sigma_{ij}^{\text{dev}} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \,. \tag{5.35}$$

Applying HOOKE's law to isotropic materials results in this:

$$\sigma_{ij} = \lambda \text{tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\varepsilon_{ij} \,. \tag{5.36}$$

When also separating the two parts of the strain tensor, HOOKE's law looks as follows:

$$\sigma_{ij} = 3\lambda\varepsilon_{ij}^{\text{vol}} + 2\mu\left(\varepsilon_{ij}^{\text{vol}} + \varepsilon_{ij}^{\text{dev}}\right) . \tag{5.37}$$

Splitting the stress in the two parts as already mentioned

$$\sigma_{ij} = \sigma_{ij}^{\text{vol}} + \sigma_{ij}^{\text{dev}} = (3\lambda + 2\mu)\,\varepsilon_{ij}^{\text{vol}} + 2\mu\varepsilon_{ij}^{\text{dev}} , \tag{5.38}$$

we reach the definition of both stresses depending on the two strains respectively as follows:

$$\sigma_{ij}^{\text{vol}} = (3\lambda + 2\mu)\,\varepsilon_{ij}^{\text{vol}} = \frac{E}{1-2\nu}\varepsilon_{ij}^{\text{vol}} \tag{5.39}$$

and

$$\sigma_{ij}^{\text{dev}} = 2\mu\varepsilon_{ij}^{\text{dev}} = \frac{E}{1+\nu}\varepsilon_{ij}^{\text{dev}} . \tag{5.40}$$

With the deviatoric and the volumetric part completely separated, it is now possible to consider the Rheological equation of state also split in two:

$$\tau_\sigma \frac{\mathrm{d}\sigma_{ij}^{\text{vol}}}{\mathrm{d}t} + \sigma_{ij}^{\text{vol}} = \frac{E}{1-2\nu}\left[\tau_\varepsilon \frac{\mathrm{d}\varepsilon_{ij}^{\text{vol}}}{\mathrm{d}t} + \varepsilon_{ij}^{\text{vol}}\right] \tag{5.41}$$

and

$$\tau_\sigma \frac{\mathrm{d}\sigma_{ij}^{\text{dev}}}{\mathrm{d}t} + \sigma_{ij}^{\text{dev}} = \frac{E}{1+\nu}\left[\tau_\varepsilon \frac{\mathrm{d}\varepsilon_{ij}^{\text{dev}}}{\mathrm{d}t} + \varepsilon_{ij}^{\text{dev}}\right] . \tag{5.42}$$

When adding the previous two equations (5.41 and 5.42) it results in one only equation in which the volumetric and the deviatoric part of the stress can get together and become the total stress again:

$$\tau_\sigma \frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}t} + \sigma_{ij} = \frac{E}{1-2\nu}\left[\tau_\varepsilon \frac{\mathrm{d}\varepsilon_{ij}^{\text{vol}}}{\mathrm{d}t} + \varepsilon_{ij}^{\text{vol}}\right] + \frac{E}{1+\nu}\left[\tau_\varepsilon \frac{\mathrm{d}\varepsilon_{ij}^{\text{dev}}}{\mathrm{d}t} + \varepsilon_{ij}^{\text{dev}}\right] . \tag{5.43}$$

Applying finite differences in order to eliminate the derivatives, it changes the previous equation (5.43) to this one:

$$\tau_\sigma \frac{\sigma_{ij} - \sigma_{ij}^0}{\Delta t} + \sigma_{ij} = \frac{E}{1-2\nu}\left[\tau_\varepsilon \frac{\varepsilon_{ij}^{\text{vol}} - \varepsilon_{ij}^{\text{vol},0}}{\Delta t} + \varepsilon_{ij}^{\text{vol}}\right] + \frac{E}{1+\nu}\left[\tau_\varepsilon \frac{\varepsilon_{ij}^{\text{dev}} - \varepsilon_{ij}^{\text{dev},0}}{\Delta t} + \varepsilon_{ij}^{\text{dev}}\right] . \tag{5.44}$$

From it, the next expression of the stresses can be deduced

$$\begin{aligned}\sigma_{ij} = &\tfrac{\tau_\sigma \sigma_{ij}^0}{\tau_\sigma + \Delta t} + \tfrac{E}{(1-2\nu)(\tau_\sigma + \Delta t)}\left[(\tau_\varepsilon + \Delta t)\,\varepsilon_{ij}^{\text{vol}} - \tau_\varepsilon \varepsilon_{ij}^{\text{vol},0}\right] \\ &+ \tfrac{E}{(1+\nu)(\tau_\sigma + \Delta t)}\left[(\tau_\varepsilon + \Delta t)\,\varepsilon_{ij}^{\text{dev}} - \tau_\varepsilon \varepsilon_{ij}^{\text{dev},0}\right] .\end{aligned} \tag{5.45}$$

In the expression above (5.45) are used the variables $\varepsilon_{ij}^{\text{vol}}$ and $\varepsilon_{ij}^{\text{dev}}$ which express the volumetric and deviatoric strains separately. These strains can be calculated as follows in Cartesian coordinates:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij}^{\text{vol}} + \varepsilon_{ij}^{\text{dev}} . \tag{5.46}$$

As for the volumetric part

$$\varepsilon_{ij}^{\text{vol}} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \frac{1}{3}\frac{1}{2}\left(2\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} = \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} , \tag{5.47}$$

from which it can be written

$$\varepsilon_{ij}^{\text{dev}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) . \tag{5.48}$$

A part from these expressions, there are also two other variables that need to be explained. With the finite differences, we distinguished between the current value of the strain and the value that it had the instant before (with a difference of time equal to Δt). These variables are written with a zero on top of them, for instance: $\varepsilon_{ij}^{\text{vol},0}$, $\varepsilon_{ij}^{\text{dev},0}$ and σ_{ij}^{0}. In the code, all this expressions are given the value of $\varepsilon_{ij}^{\text{vol}}$, $\varepsilon_{ij}^{\text{dev}}$ and σ_{ij} that had in the previous iteration.
Also in this case, the time normalization has to be done in order to know where to finish the calculations. It will be done following the expressions (5.12) and (5.13). The result in the expression of stress σ (5.45) is the following:

$$\begin{aligned}\sigma_{ij} = &\tfrac{\tau_\sigma\sigma_{ij}^{0}}{\tau_\sigma+\tau_\varepsilon\Delta\tilde{t}} + \tfrac{E}{(1-2\nu)\left(\tau_\sigma+\tau_\varepsilon\Delta\tilde{t}\right)}\left[\left(\tau_\varepsilon+\tau_\varepsilon\Delta\tilde{t}\right)\varepsilon_{ij}^{\text{vol}} - \tau_\varepsilon\varepsilon_{ij}^{\text{vol},0}\right] \\ &+\tfrac{E}{(1+\nu)\left(\tau_\sigma+\tau_\varepsilon\Delta\tilde{t}\right)}\left[\left(\tau_\varepsilon+\tau_\varepsilon\Delta\tilde{t}\right)\varepsilon_{ij}^{\text{dev}} - \tau_\varepsilon\varepsilon_{ij}^{\text{dev},0}\right] .\end{aligned} , \tag{5.49}$$

and in the rheological equation (5.44):

$$\tau_\sigma\frac{\sigma_{ij}-\sigma_{ij}^{0}}{\tau_\varepsilon\Delta\tilde{t}} + \sigma_{ij} = \frac{E}{1-2\nu}\left[\frac{\varepsilon_{ij}^{\text{vol}}-\varepsilon_{ij}^{\text{vol},0}}{\Delta\tilde{t}} + \varepsilon_{ij}^{\text{vol}}\right] + \frac{E}{1+\nu}\left[\frac{\varepsilon_{ij}^{\text{dev}}-\varepsilon_{ij}^{\text{dev},0}}{\Delta\tilde{t}} + \varepsilon_{ij}^{\text{dev}}\right] . \tag{5.50}$$

This two normalized equations (5.49) and (5.50) are the ones used in the simulation.

Relaxation times As for the values of the constants of time (τ_ε and τ_σ), both of them are going to be around values from $10^6 - 10^3$ s. This conclusion has been taken by calculations on graphics from the bibliography [8, 11].
The first of the articles show a graphic shown in Figure 5.10 in which is plotted the strain in relation with time. Knowing that

$$\tau_\varepsilon = \frac{\Delta t}{\Delta\%} = \frac{\Delta t}{\Delta\left(100\varepsilon\right)} , \tag{5.51}$$

we can take two points of the graphic where aluminum (73 MPa) is plotted and do the following calculations:

$$\tau_{\varepsilon}^{\text{aprox}} = \frac{\Delta t}{\Delta\left(100\varepsilon\right)} = \frac{\left(10^3 - 10\right)}{0.4 \cdot 10^{-3}} \approx 2.5 \cdot 10^6 \,\text{s}\,, \tag{5.52}$$

This value is approximate because the calculations have been made on a not very precise graphic and under the simplification of a constant slope equal to τ_{ε}. Moreover, the material studied in the paper is not the steel but the aluminum.

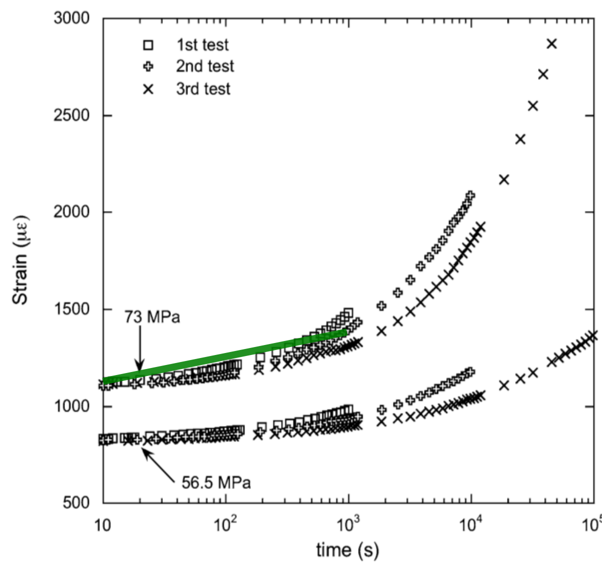


Figure 5.10: Graphics from [8, Fig. 2] relating strain and time relaxation

To enclose the value of the relaxation times we appeal to the second graphic (Figure 5.11) where the material studied is Sodium Silicate ($Na_2Si_20_5$) also known as silicon [11, Fig. 2]. This material is one of the most common in the Earth, not only in the crust, but also in the mantel. In this graphic it can be seen the relation between the temperature and the relaxation time of the material. According to a new study the Earth temperature between the inner and the outer core is around 930 K [6]. Introducing this value to the plot already referred, the relaxation time is around $10^{\mathbf{3}}$ s.

Therefore, in this study, we are going set the value of τ_{ε} between $10^{\mathbf{3}}$ and 10^6. Moreover, the values of τ_{σ} are going to be almost the same, but always a bit less, because of the thermodynamic conditions [7, §12.4.4].

5.4.3 Weak form

From the equation (5.30), which describes the balance of linear momentum on the case under study, we proceed to find the weak form by multiplying by a test function (ψ_j) and integrating both sides of the equation. Then it looks

$$\int\limits_{\Omega} \frac{\partial \sigma_{ij}}{\partial x_i} \psi_j \mathrm{d}V = - \int_{\Omega} \rho g_j \psi_j \mathrm{d}V \,, \tag{5.53}$$

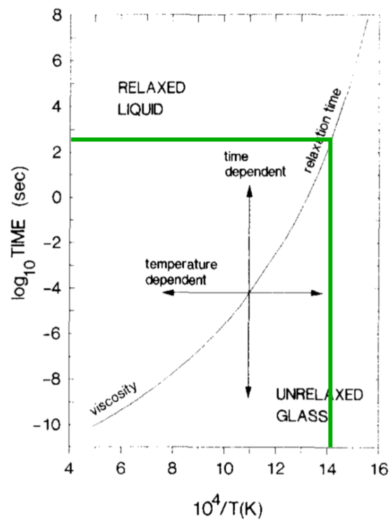


Figure 5.11: Graphics from [11, Fig. 2] relating temperature and time relaxation

and applying Gauss Theorem and partial integration we get to

$$-\int_\Omega \sigma_{ij}\frac{\partial\psi_j}{\partial x_i}\mathrm{d}V = -\int_\Omega \rho g_j\psi_j\mathrm{d}V - \oint_{\partial\Omega} n_i\sigma_{ij}\psi_j\mathrm{d}A\,. \tag{5.54}$$

In the expression above (5.54) the variable n_i stand for the normal vector of the surface of the beam. In the code, the product between the normal vector and the stress tensor is represented with a vector with the modulus and direction of the external force applied on the beam. It is called t_{ext} and it is considered as a pure tension or a pure compression along the length axis of the beam (x-axis, in our case). Then, it has the following aspect:

$$t_{\text{ext}} = \begin{bmatrix} \sigma_0^{\text{ext}} \\ 0 \\ 0 \end{bmatrix}, \tag{5.55}$$

where σ_0^{ext} is a constant value that varies depending on how much pressure is needed.
Moreover, the test function (ψ_j) is now a first order tensor (therefore, with three components) as far as the study is now in 3D.
The unknown variables to be solved with the weak form are the displacements which can be represented with a second order tensor referred as u_{ij}.
With this final form of the equation FEniCS [1] is going to be able to solve the differential equation.

5.4.4 Boundary conditions

The boundary conditions in this 3D case are also the same as in the previous cases 1 and 2 (5.2.3 and 5.3.2). The only difference is that Dirichlet boundary conditions have to be applied in tensors instead of one-dimension variables.

Therefore, the second boundary condition says that all displacements in the end where $x = 0$ have to be also 0, because it is fixed to a wall (zero degrees of freedom). The next expression sums up what is written in the code:

$$u_i|_{x=0} = 0\,. \tag{5.56}$$

Moreover, the second condition is written like this

$$\sigma_{ij} n_i|_{x=0} = 0\,. \tag{5.57}$$

5.4.5 Mesh

The 3 dimensional beam has the following measures (10,1,1) in meters in the model. Every edge is divided in a different number of slices to form tetrahedral 3D elements. The number of divisions in the mesh are (25, 8, 8) in the same directions as the axes. Therefore the number of tetrahedra and, consequently, the number of elements can be calculated as follows:

$$n_{\text{elements}} = 6 n_x n_y n_z = 6 \cdot 25 \cdot 8 \cdot 8 = 9600\,. \tag{5.58}$$

Moreover, the number of vertices is

$$n_{\text{vertices}} = (n_x + 1)(n_y + 1)(n_z + 1) = (25 + 1)(8 + 1)(8 + 1) = 2106\,. \tag{5.59}$$

Notice that the number of elements and vertices is rather high. In fact, it takes around 5 minutes to solve the whole model with this dimensions. For a more approached solution, the mesh could be refined, but it would mean the time of calculation to increase quite a bit.
In Figure 5.12 can be appreciated the shape of the actual mesh.

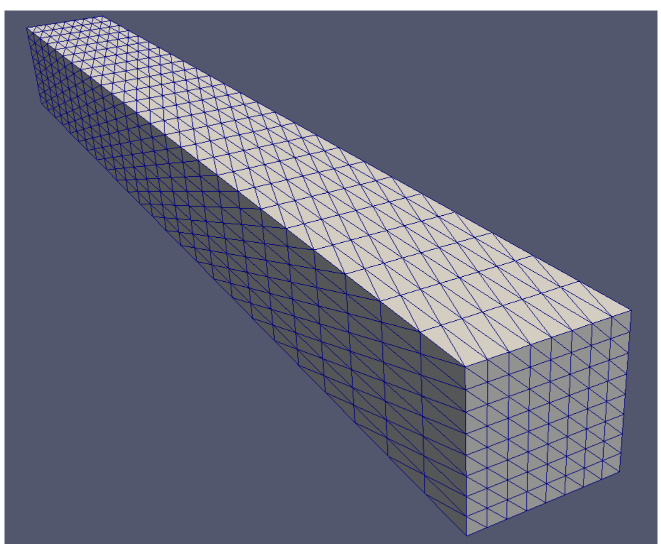


Figure 5.12: Tetrahedral mesh in 3D for the beam

5.4.6 Numerical solution

Previous solution Before solving the complete problem, the study is simplified in several steps in order to get an easier equation to be solved. It is like this because the code presented some difficulties and in order to debug it, it was easier to solve problems step by step.
In most of the times, except when it is said the opposite, the values of constants already normalized are the following:

Constant	Value
τ_σ	$1 \cdot 10^5$s
τ_ε	$1 \cdot 10^4$s
$\tilde{t}_{\text{ext}}$	$\frac{1 \cdot 10^8}{E}$
$\tilde{g}$	1

Table 5.1: Table with the exact values of each constant appearing in the study

In the previous table (Table 5.1), the value of g is equal to one when the gravitational force is considered and zero when the mass of the beam is underestimated. As fas as t_{ext} is concerned, it is also dimensionless and equivalent to a pressure of $1{\cdot}10^8$ MPa.
Moreover, in the next simplified approaches, the only variable that is changed is the expression of stress tensor (σ_{ij}). It is fair to say that the weak form (5.54) and the definition of strains (5.47 and 5.48) stay as mentioned before.

First approach: $\tau_\sigma = 0$, $\tau_\varepsilon = 0$ Under the simplification of both time constants equal to zero, the expression of the stress tensor is the following:

$$\sigma_{ij} = \frac{E}{(1-2\nu)}\varepsilon_{ij}^{\text{vol}} + \frac{E}{(1+\nu)}\varepsilon_{ij}^{\text{dev}} . \tag{5.60}$$

The solution in this case does not depend on the time and shows a linear behavior. The displacement along the length-axis (also known as x-axis) varies linearly from 0 (where the beam is being hold) until 0,00485 in the end where the external pull is applied.
The slope of the displacement depending on the x in the first approach is $slope_{\text{ap1}} = \frac{\partial \tilde{u}_1}{\partial \tilde{x}_1} = 0.00485$.

Second approach: $\tau_\sigma = 0$, $\tau_\varepsilon \neq 0$ On this second approach the expression of the stress tensor has the next appearance:

$$\sigma_{ij} = \frac{E}{(1-2\nu)}\frac{1}{\tau_\varepsilon \Delta\tilde{t}}\left[\left(\tau_\varepsilon + \tau_\varepsilon \Delta\tilde{t}\right)\varepsilon_{ij}^{\text{vol}} - \tau_\varepsilon \varepsilon_{ij}^{\text{vol},0}\right] + \frac{E}{(1+\nu)}\frac{1}{\tau_\varepsilon \Delta\tilde{t}}\left[\left(\tau_\varepsilon + \tau_\varepsilon \Delta\tilde{t}\right)\varepsilon_{ij}^{\text{dev}} - \tau_\varepsilon \varepsilon_{ij}^{\text{dev},0}\right] . \tag{5.61}$$

As can be seen in the previous expression (5.61), the solution depends also on time. Therefore, there is not only one solution, but as many solutions as we write in the code. In this case, the solution is calculated in 80 different iterations. Considering the solution already in a stationary state when arriving to its 98% of the final value, as commented before (5.2.2), thanks to time normalization.

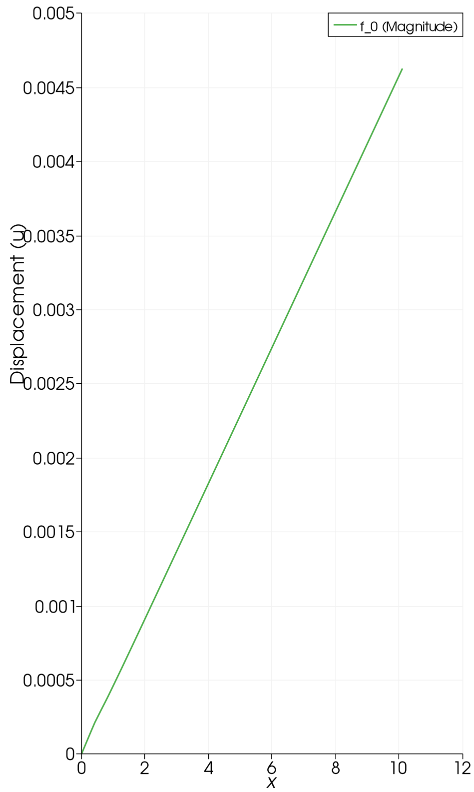


Figure 5.13: Displacement depending on the position (x) in the 2st approach in the last iteration

After the last iteration the displacement value in the end of the beam escalates until 0.00463. Even though values change between iterations, all plots of displacement in relation to x-position (length axis) in every iteration time are linear. The last solution presents the next appearance:
In this graphic (Figure 5.13) can be seen the linear distribution of displacement along the beam. From the end of the beam where it is fastened ($x = 0$), where the displacement has to be 0, until the other end ($x = 10$) of the beam, where the displacement is maximum and equal to 0.00483. When watching the evolution of the plots with the same scale in both axis, it can be seen how slopes change within the iterations. Thanks to seeing the evolution, we can notice that the slope as the time passes by are varying less in every iteration until reaching almost the stationary solution.
The previous graphic (Figure 5.14) shows the evolution of the magnitude of displacements u in two points of the beam. On the one hand we have the very end of the beam ($x = 10$) where the displacements are maximum and the mid point of the beam ($x = 5$) is also plotted.

Third approach: $\tau_\sigma \neq 0$, $\tau_\varepsilon = 0$ Please see below how the expression of the stress looks like in the case when only τ_ε equals to zero:

$$\sigma_{ij} = \frac{\tau_\sigma \sigma_{ij}^0}{\tau_\sigma + \tau_\varepsilon \Delta\tilde{t}} + \frac{E}{(1-2\nu)\left(\tau_\sigma + \tau_\varepsilon \Delta\tilde{t}\right)} \left[\tau_\varepsilon \Delta\tilde{t} \varepsilon_{ij}^{\text{vol}}\right] + \frac{E}{(1+\nu)\left(\tau_\sigma + \tau_\varepsilon \Delta\tilde{t}\right)} \left[\tau_\varepsilon \Delta\tilde{t} \varepsilon_{ij}^{\text{dev}}\right] . \quad (5.62)$$

When analyzing the previous expression (5.62) and bearing in mind that the differential of time

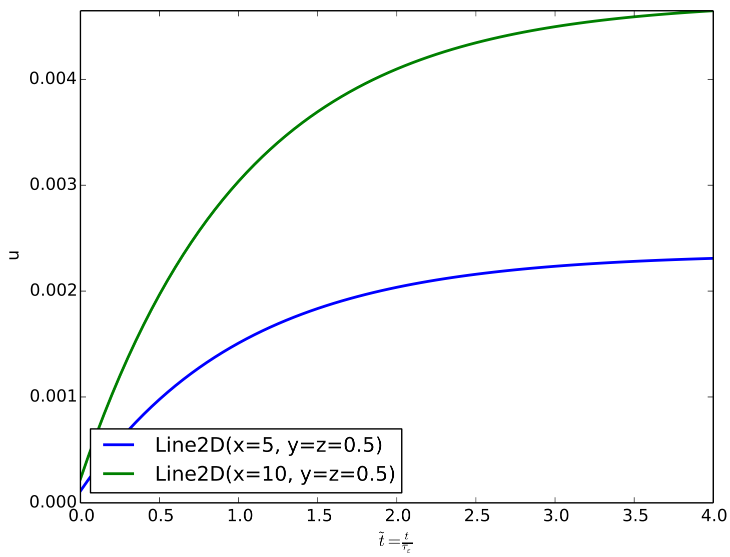


Figure 5.14: Displacement depending on time in the middle and end of the beam in the 2nd approach

multiplied by τ_ε is zero in this case, the expression turns to be almost equivalent to this one:

$$\sigma_{ij} = \frac{\tau_\sigma \sigma_{ij}^0}{\tau_\sigma} \, . \tag{5.63}$$

With it can be deduced that the σ_{ij} does not change in the process because the stress on the next step will be the same as the stress on the previous one.
Moreover, having a look at the equation of the Rheological material law (5.43), and taking into account that σ_{ij} does not change in relation with time (then, $\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}t} = 0$) we can conclude that: Strains do not variate nor even within time or position. Therefore, displacements will not change either. Then, this third approach does not make sense at all.
Actually, the only fact of τ_σ being greater than τ_ε does not make sense according to thermodynamics, because this approach does not obey the stability conditions [7, §12.4.4].
In addition, when solving the code under this hypothesis, the software does not converge. Being this another proof that confirms what said before.

Solution in x-axis With all previous steps taken, now is the time to study the complete model. It means with both values of time constants (τ_ε and τ_σ) and σ_{ij} as the expression (5.45) describes. First of all, it has to be said that all results in this subsection are taken along the x-axis and in the exact core of the beam. It means, then, that in all points the displacements are calculated in coordinates x, $y = 0.5$ and $z = 0.5$.
The magnitude of displacements are plotted in the next graphic (Figure 5.15) depending on the time. Moreover, it shows the solution in two different points of the beam, in the middle point and the free end.
In Figure 5.15 can be appreciated that the solution presents an exponential behavior because the shape describes a decreasing slope as time passes by. Moreover, notice that the solution tends to a

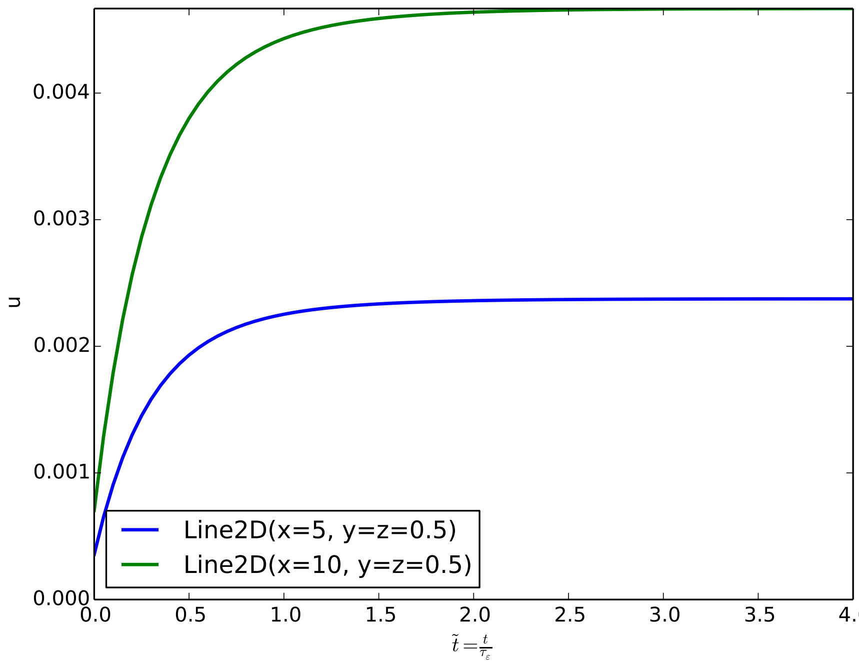


Figure 5.15: Displacement plot depending on time in two points of the 3D beam

final value of displacement equal to 0.00466 in the end of the beam ($x = 10$). As the displacement follows a linear distribution in the beam in the x-direction (length direction), the graphic also shows that the final value of the displacement when $x = 5$ is $u_{x=5} = \frac{u_{x=10}}{2} = \frac{0.00466}{2} = 0.00233$.

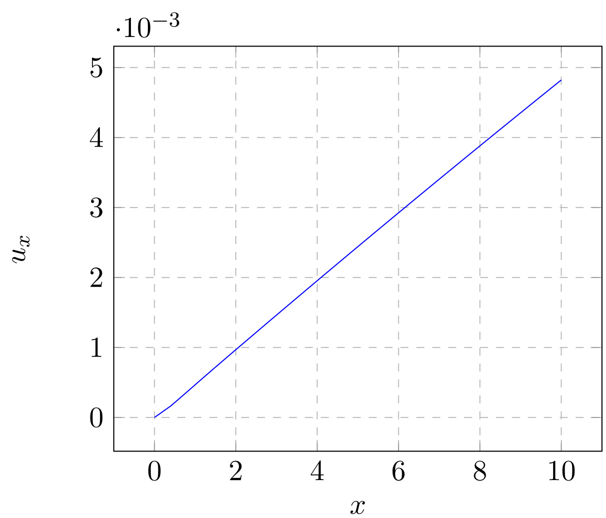


Figure 5.16: Displacement along the x-axis in the 3D beam

In Figure 5.16 can be seen the linear distribution of displacements along the x-axis in the final iteration. The final iteration happens when $\tilde{t} = 4$. Therefore, we can calculate the time when the beam is going to arrive to the stationary state as follows:

$$t = \tau_\varepsilon \tilde{t} = 4 \cdot 10^5\,\mathrm{s} \;= 111.11\,\mathrm{h}\,. \tag{5.64}$$

Then, the beam will remain static approximately when 111 hours from the application of the external force. Then the strains, stresses and displacements will remain as they are until the equilibrium is altered.
As for the strain in the x-axis, the final value as Figure 5.17 shows is 4,55·10^{-5}. Moreover, the strain also shows an exponential behavior (as expected). The initial value should be:

$$\varepsilon\big|_{t=0} = \frac{\sigma_0^{\mathrm{ext}}}{E_0}\frac{\tau_\sigma}{\tau_\varepsilon} = \frac{10^8}{210 \cdot 10^9}\frac{10^4}{10^5} = 4.762 \cdot 10^{-5}\,. \tag{5.65}$$

According to the plot (Figure 5.17) and with the help of paraview, the initial value is 6,77·10^{-5}. Therefore, there is a difference between the analytical and numerical solution in the initial value of $\delta_{\mathrm{initial}} = \frac{4.762 \cdot 10^{-5} - 6.77 \cdot 10^{-5}}{6.77 \cdot 10^{-5}} 100 = 29.7\%$.

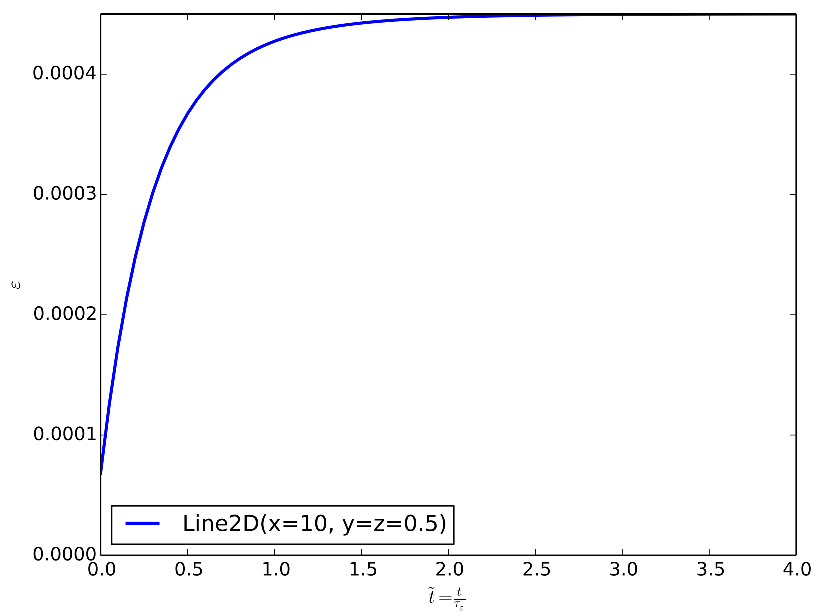


Figure 5.17: Evolution of strain in the very end of the beam along time

Solution in y- and z-axis Until now, all previous studies have been done considering only one dimension. But in this case, there are also strains, stresses and displacements in the perpendicular directions of x. In this subsection all of them are going to be analyzed.
To begin with, the displacements along the section are being analyzed in different values of x (length axis).
For example, in the following graphic (Figure 5.18) can be seen the displacements along y-axis in coordinates such as (x,y,0.5), where x takes the next values: 0.1, 0.4, and 10.

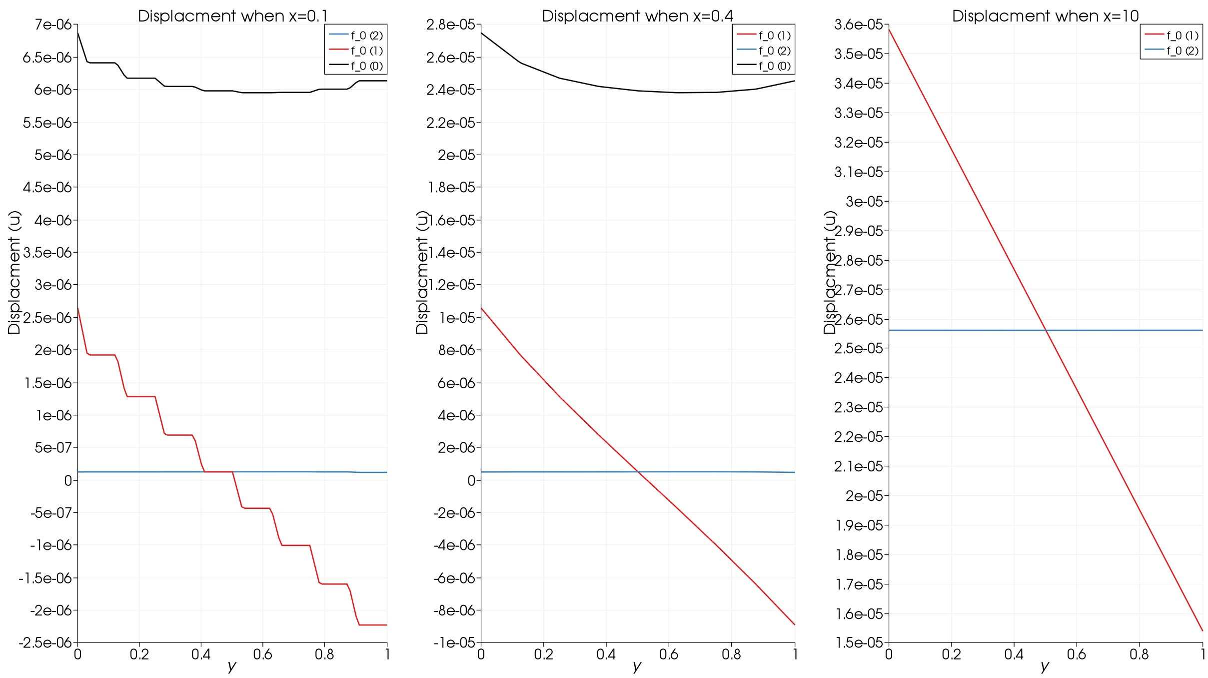


Figure 5.18: Plots of displacements along the y-axis for values x=0.1, 0.4 and 10

In Figure 5.18, there are three different colors which represent the displacement in every direction of space. First of all, the color black represents the magnitude of displacement in the x-axis. The red one represents the displacement along y-axis. Finally, the blue plot makes reference to the displacement along the z-axis.

Looking at the first plot of Figure 5.18 (when $x = 0.1$) we are able to see how many division the y axis of the mesh has. The red line (u_y) shows a plot similar to some stairs with 8 steps. Therefore, the mesh is composed by 8 divisions and 9 nodes. This "saw" shape is getting less evident as soon as x increases. In fact, in the plot representing $x = 0.4$, this stair shape almost cannot be seen. The red line is virtually linear.

But this red line should be ideally constant. The slope ($\frac{\partial u_y}{\partial y}$) of the red line should be equal to 0. Why it is not like this? This is because of the shape of the mesh. The mesh is not symmetric, therefore it results in a small slope of only $\frac{\partial u_y}{\partial y} = \frac{1.6 \cdot 10^{-5} - 3.6 \cdot 10^{-5}}{1-0} = -2 \cdot 10^{-5}$, when $x = 10$. Insisting on it, the ideal value of this slope should be 0.

This non-symmetry of the mesh can be seen in the Figure 5.19. The first figure shows that the arrows of displacement are not straight lines following strictly the x-direction. In the second picture we can see the actual shape of the mesh, in which all diagonal lines follow the same direction, instead of alternating them.

Thanks to this plots (Figure 5.18) we can see that the displacements in y and z direction are much smaller than the one in the x direction. Moreover, the difference between them increases as x also increases.

As for the displacement in the direction z, it is always constant because the measures had been taken with a constant value of z equal to 0.5.

When talking about the displacement in the x-direction (black line of Figure 5.18), we can see that they describe a parabola with a minimum near the value $y = 0.5$. It means, that in the middle of the beam, the displacement along x-axis is minimum. Nonetheless, notice that the difference between the maximum and the minimum of this parabola is equal to $3.8 \cdot 10^{-6}$ in the

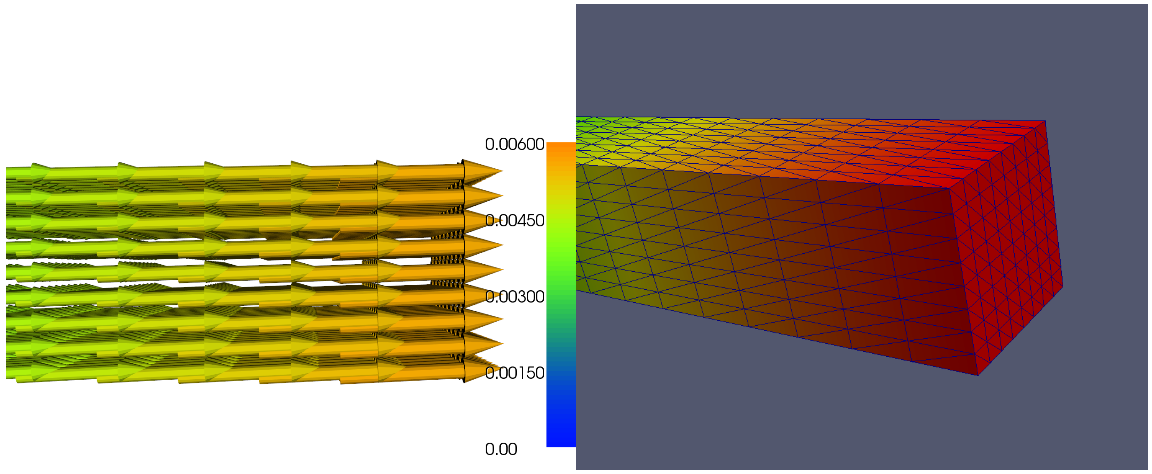


Figure 5.19: Two ways of noticing that the mesh is not symmetric

free end of the beam ($x = 10$) according to detailed information taken from paraview.
When representing the plots along the z-axis for different values of x, the plots are very similar to the ones showed in Figure 5.18. That is why there are not any graphics referring to the strains and displacement along the z-axis.
Both displacements and strains perpendicular to the x-axis are caused by POISSON's ratio. In our case, the beam's Poisson's ratio is equal to 0.3. Bearing in mind the definition of this coefficient:

$$\nu = -\frac{\varepsilon_{\text{trans}}}{\varepsilon_{\text{axial}}}, \tag{5.66}$$

we can notice that if there is an axial strain, there will be a transversal strain, too. Therefore, if there are transversal strain there will also be transversal displacements. As a consequence, the plots in Figure 5.18 are not only caused because of the "bad" mesh, but also because of physical reasons.

5.4.7 Comparison with 1D case

When comparing the two first 1D cases (5.2 and 5.3) the only aspect to compare is the stresses along the x-axis. There are two possible comparisons: when there is only constant stress applied (5.2) and when there are both constant stress and the gravitation applied (5.3). Take into account that the boundary conditions are not equivalent with the previous 1D cases. Therefore, a difference between them is being expected.

Case 1: Constant stress When simulating the 3D case with only the external force applied *i.e.* $g = 0$ the shape of epsilon depending on normalized time is the one showed in Figure 5.20. Comparing it with the plot one calculated in Case 1 (Figure 5.2) when solving the analytical problem we can see that the final value is coincident (around 0.00046), but the initial value differs a bit with the one calculated analytically (5.15).

Case 2: Constant stress and gravitational force When comparing the whole model (also with the gravitation) the x solution showed in Figure 5.17 shows a very similar shape and values as the one showed when solving case 2 numerically (see Figure 5.8 on page). In fact, the final

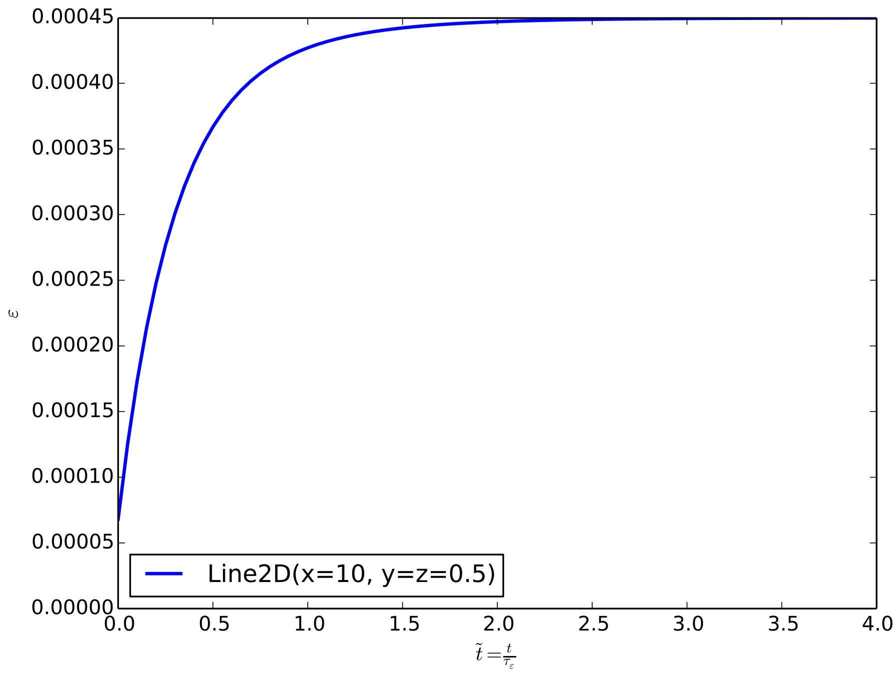


Figure 5.20: Strain evolution with time when the gravitation is not applied

value is coincident. Obtaining the same solution with two different procedures adds credibility to the simulations done.
Nonetheless, the initial value is not exactly the same, since the solution obtained in case 2 (see Figure 5.8 on page) shows an initial value of the strain of 0. According to the analytical study (5.65), the initial value of the strain should be around $5{\cdot}10^{-4}$, but not in 0. But looking the plot (Figure 5.8) around t=0, we see that on the second iteration, the value that the solution shows is around $5{\cdot}10^{-4}$. Therefore, it means that on the first iteration the strains takes an initial value of 0 but actually, the first value to take into account should be the one taken in the second iteration. This is like this, because in the code the initial value for all variables is 0, in order to make the solution converge faster.

5.4.8 Abstract

This is the first three-dimensional model that follows a material law depending on time. As seen in the numerical solution (subsection 5.4.6), both the shape and the final values of the different solutions are the ones to be expected. In fact, all solutions in the axial direction (strains and displacements) follow an exponential curve that reaches a final value in $\tilde{t} = 4$, as expected. Moreover, the final values are completely coherent with the ones obtained through other ways. For instance, on one-dimensional models or with an analytical way. Despite this, a significant mistake is noticed in the initial values of the displacement and strains.
As for the solutions in both transversal directions (y and z) there are notably smaller than the ones in the longitudinal direction. This is completely coherent with the material laws. Nonetheless, a small fraction of this transversal displacements and strains are caused by the mesh. A possible solution would be to change the mesh in order to reduce even more the deflected value. Alternating

the mesh could be done in three different ways at list:

- Change the number of divisions in the mesh
- Change the shape of the elements
- Change the direction or position of the triangles in the mesh.

5.5 Case 4: 3D Earth model under own gravitation effect

5.5.1 Introduction and Hypothesis

The Earth is considered as a perfect sphere and made out of homogeneous steel. Therefore, the values of POISSON's ratio (ν), YOUNG's modulus (E) are the same as in case 3 (5.4) and they do not vary with time. As for the density (ρ_0) we get the value of the 5515 kg/m$^{\mathbf{3}}$.
The body is considered isolated and resting. Therefore, there are no accelerations taking place in order not to complicate the model excessively. The spin of the Earth is neglected according to the results obtained in the example located in subsection 3.2.1. As a result, the only stress suffered by the Earth is the self gravitation caused by its own mass.
As for the material law, the solution is calculated following the viscoelastic description of materials. In particular, the equation used to describe the behavior of the material is the Rheological equation for 3D cases and already normalized by time (5.50). In this expression (5.50) we see that linear strain is used ($\boldsymbol{\varepsilon}$). Therefore, it is being assumed that displacements are going to be infinitesimal. Although treating a huge object such as a planet under big stresses, we consider the state of deformation as infinitesimal.
The values of relaxation time constants (τ_ε and τ_σ) are the same calculated in the previous case (paragraph 5.4.2).

5.5.2 Variables and Equations

As in the previous cases, we depart from the balance of linear momentum (4.14). Applying the hypothesis detailed before (5.5.1) the equation results in:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\rho f_{\mathrm{i}}\,, \tag{5.67}$$

According to POISSON's equation and considering the sphere as a point of mass:

$$f_i = \frac{\partial U}{\partial x_i} \text{ and } \frac{\partial^2 U}{\partial x_i^2} = 4\pi G \rho_{\mathrm{E}}\,, \tag{5.68}$$

Nonetheless, in subsection 3.2.1 this potential is obtained numerically solving the following weak form:

$$-\int_\Omega \frac{\partial U}{\partial x_i}\frac{\partial \delta u}{\partial x_i}\mathrm{d}V = -\int_\Omega \delta u \mathrm{d}V\,, \tag{5.69}$$

and the solution to this problem is the one used as potential depending on the already normalized x ($U(x)$). Notice that δu is the test function in this case.
As for the expression of sigma in equation (5.67), the one used is the same as in case 3 of a 3D beam (5.49) already normalized by time.

5.5.3 Weak form

The weak form that results when integrating both sides of the equation (5.67) and also multiplying by the test function (ψ_i) is:

$$\int_\Omega \frac{\partial \sigma_{ij}}{\partial x_i} \psi_j \mathrm{d}V = -\int_\Omega \frac{\partial U}{\partial x_i} \delta\psi_i \mathrm{d}V\,. \tag{5.70}$$

Notice that the term $\frac{\partial U}{\partial x_i}$ stand for the gradient of the potential calculated also numerically.

5.5.4 Boundary conditions

There is one boundary conditions set in the model. The first one is set on the displacement in the core, which has to be 0 in all the simulation. Therefore, it can be written down as follows:

$$u|_{(0,0,0)} = 0\,. \tag{5.71}$$

The second one is referred to the stresses in the crust:

$$\sigma_{ij} n_j|_{\partial V} = 0\,. \tag{5.72}$$

5.5.5 Mesh

The mesh used to find the solution is a sphere of unitary radius. The center is set in the core of the sphere. In order not to have to many nodes and elements and therefore spend to much time in calculations, the radius has been divided in ten sections. The mesh tool used does the divisions in tetrahedral shape in relation with the number of division of the radius. In this case, the number of elements is 7941.
Moreover, in order to set the boundary condition in the center, it was created a domain containing the closest elements near the center of the sphere instead of considering only the central point itself.

5.5.6 Numerical solution

The spherical surface where the strain is maximum is around 0.8. The final value of the strain at this point reached after $\tilde{t} = 4$ (or also equivalent to 111.11 h, according to equation 5.64) is $\varepsilon|_{\tilde{t}=4,\,(0.8,0,0)}$ =$3.50 \cdot 10^{-2}$. We cannot compare this value to anything since the analytical solution is not so easy to find as in previous cases.
At least we can say that the initial value of the strain is not equal to 0, which matches to the material law followed. Once again, knowing $\varepsilon|_{\tilde{t}=0,\,(0.8,0,0)} = 8.02 \cdot 10^{-3}$ does not give us the certainty of having found the appropriate values.
In order to compare values with something already known, we plot the displacements in different points of the sphere along time in Figure 5.21.
In the graphic shown in Figure 5.21, all three plots show an exponential shape. However, displacement depends not only in time but also in the position.
The initial value for the displacements in the sphere depends on the position, but in any case (except for in the crust and in the core) the displacement is different to 0. Nonetheless, this initial value cannot be compared because in section 4 all analysis of results is done not depending on the time.

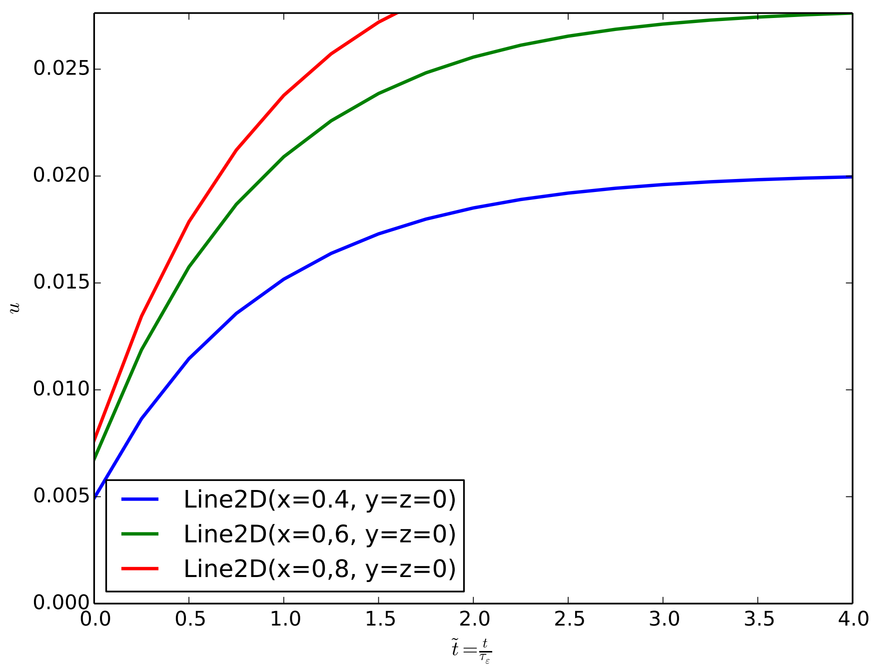


Figure 5.21: Displacement in different points of the sphere along time

As for the final value of the displacement, we see that the maximum value of the displacement is 0.0317. This displacement is already normalized by the radius since it is equal to 1.
In order to get more information about the displacements, the plot in Figure 5.22 show the displacement along the x-axis.

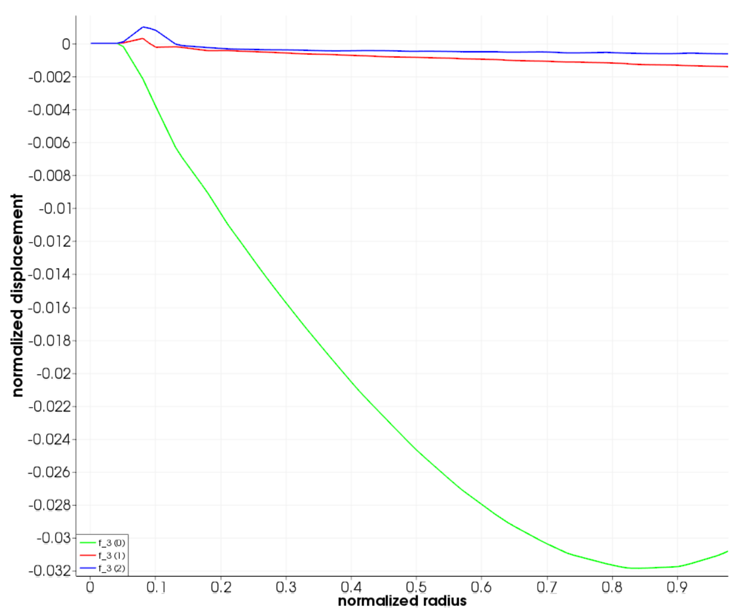


Figure 5.22: Displacements along x-axis

First of all, lets make clear what the different colors mean in Figure 5.22. Both colors red and blue represent the displacement in both normal directions of x. And color green stands for the displacement in the x-direction.
Said that, we notice that displacement in the perpendicular plane to the x-axis is almost 0. Therefore, all displacements follow the radial direction, as the theory could preview. If all the stress is along the radial axis, the displacements too. When saying these perpendicular displacements are almost 0, we associate a numerical error to the word "almost". The maximum displacement in a perpendicular direction to the radius is $3.1 \cdot 10^{-4}$, whereas the maximum displacement in the radial direction is almost 100 bigger. This way, we consider this number as a result of the numerical solution.
But the important plot is the one associated to the displacement in the radial direction (in green in the plot 5.22). The value of displacements in the core is 0, responding to the boundary conditions. As for the shape, the displacement shows a minimum when 0.85 (therefore when $r = 0.85$). According to LOVE's theory (3.1.2), the radius where the minimum is located should be:

$$r = r_0 \left(\frac{3 - \nu}{3 + 3\nu} \right)^{\frac{1}{2}} = 1 \left(\frac{3 - 0,3}{3 + 3 \cdot 0,3} \right)^{\frac{1}{2}} = 0.832\,. \tag{5.73}$$

As a consequence, the model follows the LOVE theory.
Moreover, at this surface ($r = 0.85$) the minimum value of displacement is -0.0317 in the radial direction. It means that the sphere shrinks (negative sign) only 3.17% of the initial radius according to this model.
Comparing this values to the ones found when analyzing the sphere in the current placement in the u-formulation (subsection 4.1) we see a huge differences:

- In subsection 4.1 when analyzing analytically the solution for an α equivalent to the one of the Earth, the maximum magnitude of the displacement was 0.14 whereas in this case it is only around 0.03.

In Figure 5.23 is plotted the analytical solution of the displacements along radial axis for the case of the Earth, according to the current placement calculations (subsection 4.1).
The reader can find the code corresponding to this solution in the appendix (section A.3) for more detail.

5.5.7 Abstract

The fact of finding a way to make the solution converge is always comforting, but the importance is in the solution itself. In this case, the values shown by the solution once compared with the analytical ones let the reader conclude that there are not accurate enough because:

- Displacements are 4.4 times lower than according to the analytical solution in current configuration.
- Displacements have same shape as in the analytical solution.
- Position of minimum between 0.8 and 0.9. Coherent with LOVE's theory.

Therefore, the solution found concludes that the Earth shrank with the time. Furthermore, it is caused by the gravitation created by the mass of the Earth itself and it does not depend much on the spin around its axis.

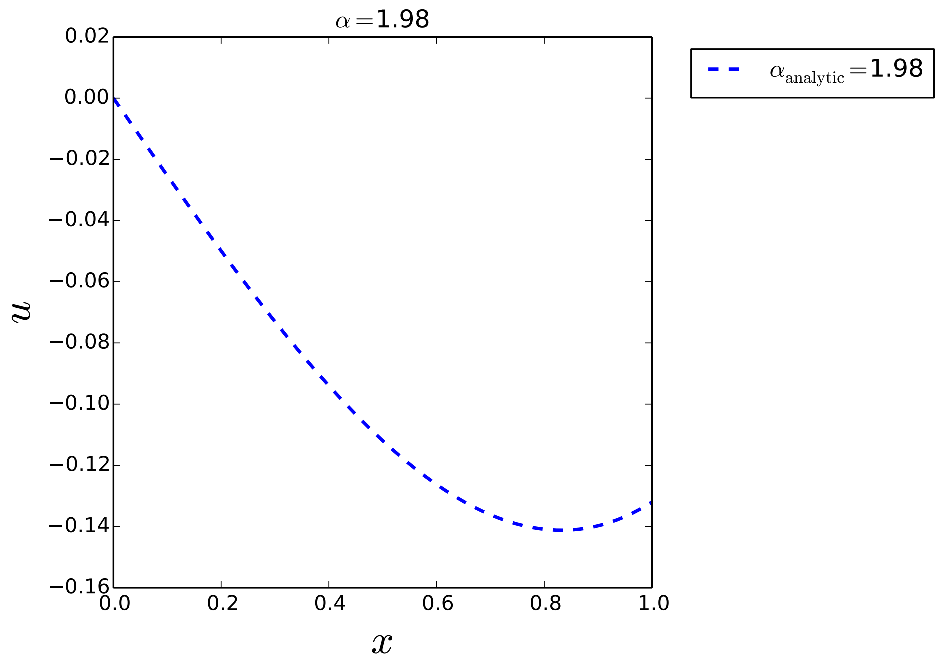


Figure 5.23: Analytical displacements along radial axis for the case of the Earth

6 Conclusion

In the course of the bachelor's thesis all secondary objectives have been achieved. Both continuum mechanics and mathematical concepts have been applied while solving all of the previous and simplified cases. Moreover, the software used (FEniCS) have been used adequately.
About the main objective of the thesis which is performing an analytical study of terrestrial objects has been achieved but with several contradictions referring to the values of the displacements. Both solutions of the Earth show the same shape but with different values of displacement.
In order to continue with the numerical study of deformation in terrestrial objects it would need to be done the same analysis but with spherical coordinates. With it, the stress tensor would be diagonal and all calculus should be easier to solve numerically.
Last but not least, LOVE's theory has been proven both in the current configuration and in the viscoelastic model, since both showed a plot of displacements with a minimum around values 0.8, which is the value LOVE predicts.

References

[1] *FEniCS Project*. http://fenicsproject.org.

[2] A. Bertram. *Elasticity and Plasticity of Large Deformations*. 2n edition edition, 2008.

[3] B. A. Bolt. Earth models with continuous density distribution. 1956.

[4] B.R. Seth and M.A. D.Sc. Some problems of finite strain. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, XXVIII, 1939.

[5] C. L. Phillips and H. T. Nagle. *Sistemas de control digital: análisis y diseño*. 2 ed. Barcelona: Gustavo Gili, 1993.

[6] E.Howell. *Earth's Core 1,000 Degrees Hotter Than Expected*. http://www.livescience.com/29054-earth-core-hotter.html, 25 May 2013.

[7] I. Müller and W. H. Müller. *Fundamentals of Thermodynamics and Applications*. Springer Verlag, 2008.

[8] J. C. Dandrea and R. L. Creep and creep recovery of cast aluminum alloys. *Springer Science*, 2009.

[9] S. C. Bose and T. T. Chattarji. A note on the finite deformation in the interior of the Earth. 1962.

[10] S. K. Pan. Finite deformation in the interior of the Earth. *University of Calcutta, Department of Mathematics*, 1962.

[11] S. L. Webb. Shear, volume, enthalpy and structural relaxation in silicate melts. 1991.

[12] W. H. Müller. *An Expedition to Continuum Theory*. Springer, 2014.

[13] W.H. Müller and P. Lofink. The movement of the Earth: Modeling the flattening parameter. *Lecture Notes of TICMI*, 15:1 – 35, 2014.

[14] Wikipedia, https://en.wikipedia.org/wiki/Dirichlet_boundary_condition. *Dirichlet boundary condition*, 2015.

[15] Wikipedia, https://en.wikipedia.org/wiki/Lam%C3%A9_parameters. *Lamé parameters*, 2015.

[16] Wikipedia, https://en.wikipedia.org/wiki/Neumann_boundary_condition. *Neumann Boundary Condition*, 2015.

[17] Wikipedia, https://en.wikipedia.org/wiki/Newton%27s_law_of_universal_gravitation. *Newton's law of universal gravitation*, 2015.

[18] Wikipedia, https://en.wikipedia.org/wiki/Newton%27s_laws_of_motion. *Newton's laws of motion*, 2015.

A Appendix

A.1 Code from the u-formulation in the current configuration (4.1):

```
from dolfin import *
import numpy as np
import pylab as plt

resolution = 1000
print 'resolution: ', resolution

mesh = IntervalMesh( resolution, 0. ,1.)
tol = 0.0001 #tolerance for coordinate comparisons

class LeftBoundary(SubDomain):
   def inside(self, x, on_boundary):
      return on_boundary and abs(x[0]) < tol
class RightBoundary(SubDomain):
   def inside(self, x, on_boundary):
      return on_boundary and abs(x[0] -1) < tol

edge = FacetFunction('size_t', mesh)
edge.set_all(0)
domain = CellFunction('size_t', mesh)
domain.set_all(0)
LeftBoundary().mark(edge,1)
RightBoundary().mark(edge,2)

file = File('plot_edge_u.pvd')
file << edge

X = FunctionSpace(mesh, 'CG', 1)
x = Expression('x[0]')

bc= DirichletBC(X, Constant(0.), edge, 1)

u = Function(X)
delu = TestFunction(X)

up = u.dx(0)
G = 6.673E-11
g = 9.80665           # m/s^2
nu = 0.3
E = 210.0E9           # Pa
rho0 = 5515.0         # kg/ m^3
a = 6378137.0         # m
mu = E/(2.*(1.-2.*nu))
lam = E*nu/((1.+nu)*(1.-2.*nu))
K = E/((1.-2.*nu)*3.)
alpha = 4.*pi*G*(rho0**2.)*(a**2.)/(3.*K)#

sigrr = 3. * ( 1. - nu) / (1. + nu) * up * (1. - 0.5* up) + 6.*nu/ (1. +nu )*u/x *
    (1.-0.5*u/x)
sigthth = 3.*nu / (1. + nu) * up * (1. - 0.5*up) + 3./ (1. +nu )*u/x * (1.-0.5*u/x)
```

```
rhs = alpha*(1.-up) * (1.-u/x)**5*x

linew = 2
fontx = 24

alphas = [0.1,0.2,0.3,0.7,1.,1.1,1.2,1.3]
x_achse = plt.linspace(0,1., resolution)
u_achse = plt.linspace(0,1., resolution)
u_ana = plt.linspace(0,1., resolution)
u_bor = plt.linspace(0,1., resolution)

fig = plt.figure()
ax = plt.subplot(1,1,1)

lines = []
colors = ['b','g','r','c','m','y','k','burlywood']

k = 0
for alpha in alphas:
   info_green(str(alpha))
   x = Expression('x[0]')
   delu = TestFunction(X)
   sigrr = 3. * ( 1. - nu) / (1. + nu) * u.dx(0) * (1. - 0.5* u.dx(0)) + 6.*nu/ (1.
       +nu )*u/x * (1.-0.5*u/x)
   sigthth = 3.*nu / (1. + nu) * u.dx(0) * (1. - 0.5*u.dx(0)) + 3./ (1. +nu )*u/x *
       (1.-0.5*u/x)
   rhs = alpha*(1.-u.dx(0)) * (1.-u/x)**5*x
   lhs = -sigrr *delu.dx(0)*dx + (2./x)*sigrr*delu*dx-(2./x)*sigthth*delu*dx - rhs
       *delu*dx
   '''solve(lhs==0, ur, bc,
       solver_parameters={'newton_solver':{'linear_solver':'mumps'}})'''
   solve(lhs == 0, u, bc, \
        solver_parameters={"newton_solver":{"linear_solver": "gmres",
            "relative_tolerance": 1E-5, "absolute_tolerance": 1E-5,
            "maximum_iterations": 5, "error_on_nonconvergence":False} }, \
        form_compiler_parameters={"cpp_optimize": True, "representation": "quadrature",
            "quadrature_degree": 2} )
   class UAnaExpr(Expression):
      def eval(self, value, x):
         value[0] = -alpha*((1.+nu)/(1.-nu))/30.0 * ( (3.0-nu)/(1.0+nu) - x[0]*x[0] ) *
             x[0]
      def value_shape(self):
         return (1,)
   u_analytic=UAnaExpr()
   for i in range(x_achse.size):
      u_achse[i] = u(x_achse[i])
      u_ana[i] = u_analytic(x_achse[i])
   ax.plot(x_achse,u_achse,color=colors[k],linewidth=linew,label=r'$\alpha=$'+str(alpha))
   ax.plot(x_achse,u_ana,linestyle='--',color=colors[k],linewidth=linew,label=r'$\alpha_{\mathrm{anal
   k+=1

   ax.set_xlabel(r'$\frac{r}{a}$')
   ax.set_ylabel(r'$\frac{u}{a}$')
```

```
   ax.set_title(r'$\alpha=$'+str(alpha))
   ax.xaxis.label.set_fontsize(fontx)
   ax.yaxis.label.set_fontsize(fontx)
   handles, labels = ax.get_legend_handles_labels()
   ax.legend(handles,labels, bbox_to_anchor=(1.05, 1), loc=2)

   fig.subplots_adjust(left=0.15)
   fig.subplots_adjust(right=0.7)
   fig.subplots_adjust(bottom=0.15)

   fig.savefig('1d_alpha_'+str(alpha)+'hola.eps')
   fig.savefig('1d_alpha_'+str(alpha)+'hola.pdf')

fig.show()
fig.waitforbuttonpress()
```

A.2 Code from the β-formulation in the current configuration (4.2):

```
from dolfin import *
import pylab as plt

resolution = 1000
mesh = IntervalMesh( resolution, 0. ,1.)
tol = 0.001 #tolerance for coordinate comparisons

class LeftBoundary(SubDomain):
   def inside(self, x, on_boundary):
      return on_boundary and abs(x[0]) < tol
class RightBoundary(SubDomain):
   def inside(self, x, on_boundary):
      return on_boundary and abs(x[0] -1.0) < tol

edge = FacetFunction('size_t', mesh)
edge.set_all(0)
domain = CellFunction('size_t', mesh)
domain.set_all(0)
LeftBoundary().mark(edge,1)
RightBoundary().mark(edge,2)

file = File('edge_beta.pvd')
file << edge

X = FunctionSpace(mesh, 'CG', 1)
x = Expression('x[0]')

bc= DirichletBC(X, Constant(0.), edge, 1)

G = 6.673E-11
g = 9.80665           # m/s^2
nu = 0.3
E = 210.0E9           # Pa
rho0 = 5515.0         # kg/ m^3
```

```
aR = 6378137.0          # m
mu = E/(2.*(1.-2.*nu))
lam = E*nu/((1.+nu)*(1.-2.*nu))
K = E/((1.-2.*nu)*3.)
alpha = 4.*pi*G*(rho0**2.)*(a**2.)/(3.*K)
#alpha = 0.27991*2./3. * (1.+ nu) / (1. - nu)
print "alpha: ", alpha

dummy = interpolate(Expression('alpha/30. * (1.+nu) / (1. - nu) * ( (3. - nu) / (1. +
    nu) - pow(x[0],2))+1', alpha=alpha, nu=nu),X)
beta = Function(dummy)

delbeta = TestFunction(X)
betap = beta.dx(0)

sigrr = 3./2. * (1.-beta**2) - 3./2. * ((1.-nu)/(1.+nu)) * x**2*(betap**2 +
    2.*beta*betap/x)
sigthth = - 3.*nu/2./(1.+nu) * x**2 * (betap**2 + 2.*beta*betap/x) + 3./2.*
    (1.-beta**2)
fr = alpha * (beta + betap * x ) * beta**5 * x
ds = ds[edge]
lhs = -sigrr *delbeta.dx(0)*dx - 3./2.* (1.-beta**2) * delbeta * ds(1) +
    betap*delbeta*ds(1)+ (2./x)*sigrr*delbeta*dx-(2./x)*sigthth*delbeta*dx
    -fr*delbeta*dx

solve(lhs == 0, beta, \
       solver_parameters={"newton_solver":{"linear_solver": "gmres",
           "maximum_iterations": 20, "error_on_nonconvergence":False} }, \
      form_compiler_parameters={"cpp_optimize": True})#, "representation":
          "quadrature", "quadrature_degree": 2} )

sigmFunc = project(sigrr,X)

# Plots
fig = plt.figure()
ax = plt.subplot(1,1,1)

linew = 2
fontx = 24
lines = []
colors = ['b','g','r','c','m','y','k','burlywood']

x_achse = plt.linspace(0,1., resolution)
b_fem = plt.linspace(0,1., resolution)
b_ana = plt.linspace(0,1., resolution)

beta_Ana = interpolate(Expression('alpha/30. * (1.+nu) / (1. - nu) * ( (3. - nu) / (1.
    + nu) - pow(x[0],2))+1', alpha=alpha, nu=nu),X)

for i in range(x_achse.size):
   b_fem[i] = beta(x_achse[i])
   b_ana[i] = beta_Ana(x_achse[i])

ax.plot(x_achse,b_fem,color=colors[0],linewidth=linew,label=r'beta_fem')
```

```
ax.plot(x_achse,b_ana,color=colors[1],linewidth=linew,label=r'beta_ana')
ax.set_xlabel(r'$\frac{r}{a}$')
ax.set_ylabel(r'$\beta$')

ax.xaxis.label.set_fontsize(fontx)
ax.yaxis.label.set_fontsize(fontx)
handles, labels = ax.get_legend_handles_labels()
ax.legend(handles,labels, bbox_to_anchor=(1.05, 1), loc=2)
fig.subplots_adjust(left=0.15)
fig.subplots_adjust(right=0.7)
fig.subplots_adjust(bottom=0.15)

fig.savefig('beta_'+str(alpha)+'.eps')
fig.savefig('beta_'+str(alpha)+'.pdf')

fileresult = File('resultbeta.pvd')
fileresult << beta
```

A.3 Code corresponding to the 3D viscoelastic Earth (5.5):

```
from dolfin import *
import numpy as N
import pylab as plt
import math

def magn(vector):
    return math.sqrt(vector[0]**2+vector[1]**2+vector[2]**2)

mesh = Mesh("../mesh/mesh4.bis.xml")
File('mesh.pvd') << mesh

class U0_boundary(SubDomain):  # Erdkruste / Erdrand
   def inside(self,x, on_boundary):
      return on_boundary

class U1_boundary(SubDomain):  # Erdkern / Erdmitte
   def inside(self,x, on_boundary):
      #tol = 5.0E5*scale_factor #highres mesh
      tol = 0.01 #lowRes mesh
      r = sqrt(x[0]*x[0] + x[1]*x[1] + x[2]*x[2])
      return abs(r)<tol

## define material values ##
G = 6.673E-11
aR = 6378137.0   # m
E = 210.0e9      # Pa
E0 = E
#mass = 5.9721986  # kg
rho0 = 5515.0    # kg/ m^3
rho = rho0
nu = 0.3
K = E/(1.-2.*nu)/3.0
```

```
alpha = 4.*pi*G*rho0**2*aR**2/3./K
omega0 = 2.*pi*1./ (24.*60.*60.)

lamda = E*nu/(1.-2.*nu) / (1.+ nu)
mu = E/(2.*(1.-2.*nu))

#Initialisierung Iteration
rand = FacetFunction("size_t",mesh)
rand.set_all(0)
domain = CellFunction("size_t",mesh)
domain.set_all(0)
U0_boundary().mark(rand,1) # Erdrand
U1_boundary().mark(rand,2) # Erdmitte

print U0_boundary().mark(rand,1), domain.set_all(0)

file = File('rand.pvd')
file << rand

S = FunctionSpace(mesh, "CG", 1)
V = VectorFunctionSpace(mesh, "CG", 1)

dA = Measure("ds")[rand]
dV = Measure("dx")[domain]

U = TrialFunction(S)
delU = TestFunction(S)

rho = 1.

U0 = Expression("-1./3.")

bc = DirichletBC(S, U0, rand, 1)

lhs = -U.dx(i)*delU.dx(i) * dV(0)
rhs = rho*delU*dV(0)

delta = Identity(3)
i,j,k,l = indices(4)

res = Function(S)
solve(lhs==rhs, res, bc, solver_parameters={'linear_solver':'mumps'})

grav = project(-grad(res),V)

file_out = File("potential2.pvd")
file_out << res
file1 = File("grav2.pvd")
file1 << grav

#---------------------------------------------------

S = FunctionSpace(mesh, "CG", 1)
V = VectorFunctionSpace(mesh, "CG", 1)
```

```
T = TensorFunctionSpace(mesh, "CG", 1)

dA = Measure("ds")[rand]
dV = Measure("dx")[domain]
                                                  #atention!!
u = Function(V)
delu = TestFunction(V)

u0 = Function(V)
u_init = Expression(('0.','0.','0.'))
u0.interpolate(u_init)

sigma0Tensor = as_tensor( u0[i].dx(j), (i,j))

bc2 = DirichletBC(V, u_init, rand, 1)  #Dirichlet conditions

te = 1e5    # strain (epsilon) time constant
ts = 1e4  # stress (sigma) time constant
# te > ts (according to thermodynamics)

delt = 0.25 #finite change of time (normalized)
t = 0.0    # initial time (t0=0)
tend = 4.

epsilon = as_tensor( 0.5 * (u[i].dx(j) + u[j].dx(i)) , (i,j))
epsvol = as_tensor( 1./3. * u[k].dx(k) * delta[i,j] , (i,j) )
epsvol0 = as_tensor( 1./3. * u0[k].dx(k) * delta[i,j] , (i,j) )
epsdev = as_tensor( 1./2. * ( u[i].dx(j) + u[j].dx(i) - 2./3. * u[k].dx(k)*delta[i,j]
    ) , (i,j) )
epsdev0 = as_tensor( 1./2. * ( u0[i].dx(j) + u0[j].dx(i) - 2./3. *
    u0[k].dx(k)*delta[i,j] ) , (i,j) )

sigma0Tensor = as_tensor( u0[i].dx(j), (i,j))

sigma = as_tensor( (ts*sigma0Tensor[i,j] / (ts + te*delt) )
     + 1./ (1. - 2. * nu) / (delt*te + ts) * ( te *(delt+1.) * epsvol[i,j] -
         te*epsvol0[i,j] )
     + 1. / (1. + nu) / (delt*te + ts) * ( te *(1.+delt) * epsdev[i,j] -
         te*epsdev0[i,j] ) , [i,j])

fig = plt.figure()
ax = plt.subplot(1,1,1)

resolution = round(tend/delt,0)+1
time = plt.linspace(0,tend, resolution)
uplot = plt.linspace(0,tend, resolution)
uplot2 = plt.linspace(0,tend, resolution)
epsplot = plt.linspace(0,tend, resolution)

count = 0

fileu= File("finalu/resultu.pvd")
file_out = File("finalu/resulteps.pvd")
fileeps = File("finalu/epsiloneps.pvd")
```

```
filestress = File("finalu/stresseps.pvd")

while t<= tend:
   info_green(str(t))
   lhs1 = 0.5*sigma[i,j]*delu[j].dx(i)*dV(0) + 0.5*sigma[i,j]*delu[i].dx(j)*dV(0)
   lhs1 += -grav[i]*delu[i]*dV(0)
   solve(lhs1==0, u, bc2)
   sigma0Tensor= project(sigma,T)
   u0.assign(u)
   fileu << u
   unova = project(u,V)
   epsnova = project(epsilon, T)
   filestress << sigma0Tensor
   t +=delt
   uplot[count] = magn(unova(0.4,0.,0.) )
   uplot2[count] = magn(unova(0.6,0.,0.) )
   epsplot[count] = magn(unova((0.8,0.,0.)) )
   fileeps << epsnova
   count += 1

ax.plot(time,uplot,linewidth=2,label=r'x=0.4, y=z=0' )
ax.plot(time,uplot2,linewidth=2,label=r'x=0,6, y=z=0' )
ax.plot(time,epsplot,linewidth=2,label=r'x=0,8, y=z=0' )

ax.set_xlim([0.,tend])
maxvalue = uplot2.max()
ax.set_ylim([0.,maxvalue])
ax.set_xlabel(r'$\tilde{t}=\frac{t}{\tau_\varepsilon}$')
ax.set_ylabel(r'$u$')
handles, labels = ax.get_legend_handles_labels()
ax.legend(handles,loc=3)

fig.show()
fig.savefig('finalu/u_over_time_.pdf')
```